On Utility of Wealth Maximization

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Abstract

This project is concerned with the continuous-time portfolio choice problem, also known as Merton's problem, when the opportunity set is stochastic (e.g. when the interest rate and/or volatility is stochastic). There are two main approaches for solving continuous-time portfolio problems: the classical stochastic control approach and the so called martingale approach. The main contribution of this project is to develop a new approach, called the *direct approach*. Unlike the stochastic control approach, it is not based on the Markov state assumption and can be extended to the general semimartingale market (though we have not tried to do so). Its advantage over the martingale approach is that the direct approach, as its name suggests, is dealing with the primal problem *directly*. So, unlike the martingale approach, the completeness or incompleteness of the market has not so much affect on it. Furthermore we are able to obtain the general form of the optimal portfolio policy directly, and not through a dual problem.

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Chapter 1

Introduction*

A fundamental concern for investors is the problem of portfolio optimization when they trade between a risk-free asset and a large number of risky assets. One of the most important decisions many people face is the choice of a portfolio of assets for retirement savings. These assets may be held as a supplement to defined-benefit public or private pension plans; or they may be accumulated in a defined contribution pension plan, as the major source of retirement income. In either case, a dizzying array of assets is available. Institutional investors also face complex decisions. Some institutions invest on behalf of their clients, like pension funds and insurance companies. Others, such as foundations and university endowments, are similar to individuals in that they seek to finance a long-term stream of discretionary spending. The investment options for these institutions have also expanded enormously since the days when a portfolio of government bonds was a norm.

Modern finance theory is often thought to have started with the mean-variance analysis of Markowitz in the early 1950s (see [63]). This made the portfolio choice theory the original subject of modern finance. He introduced the mean-variance as a criterion for portfolio selection. It is the criterion mostly used by practitioners in spite of its unrealistic hypothesis such as normal returns, single period, etc.

The continuous-time portfolio problem has its origin in the pioneering work of Merton [64, 65]. It is concerned with finding the optimal investment strategy of an investor. In other words, how much of which security she should hold at every time instant between now and a time horizon T, to maximize her expected utility from intermediate consumption and accumulated wealth at the end of the time horizon. In the classical Merton problem the investor can allocate her money into a risk-less savings account and d different risky stocks. Using the methods of stochastic optimal control, Merton derived a nonlinear partial differential equation (Bellman equation) for the value function of the optimization problem. He also produced the closed-form solution of this equation for the special cases of power, logarithmic, and exponential utility functions. A drawback of his approach, however, was the assumption of a constant investment opportunity set (i.e. constant or deterministic factors). This made the model unrealistic, specifically for long-term investors.

There are two main approaches for solving continuous-time portfolio problems: the classical stochastic control approach and the so called martingale approach. The stochastic control approach, which has been used since the seminal work of Merton, is based on the requirement of Markov state processes and involves solving a highly nonlinear PDE, the so called

*This literature study is partly excerpted from Zariphopoulou [87].

Hamilton-Jacobi-Bellman (HJB) equation. The strength of this approach is in its access to the well-developed theory of PDEs and the corresponding numerical techniques. Using this approach, the cases of special utilities (namely, the exponential, power and logarithmic) have been extensively analyzed. In these cases, convenient scaling properties reduce the associated HJB equation to a more tractable quasilinear one. However, when the utility function is general, very little is known about the maximal expected utility as well as the form and properties of the optimal policies once the log-normality assumption is relaxed and correlation between the stock and the factor is introduced. This is despite the Markovian nature of the problem at hand, the advances in the theories of fully nonlinear PDEs and stochastic control, and the computational tools that exist today. Specifically, general results on the validity of the Dynamic Programming Principle, regularity of the value function, existence and verification of optimal feedback controls, representation of the value function and numerical approximations are still lacking.

A more recent approach to the problem of expected utility maximization, which permits us to avoid the assumption of Markov asset prices, is based on duality characterizations of portfolios provided by the set of martingale measures. For the case of a complete financial market, where the set of martingale measures is a singleton, this *martingale methodology* was developed by Pliska [72], Cox and Huang [24, 25] and Karatzas, Lehoczky and Shreve [46]. Considerably more difficult is the case of incomplete financial models. It was studied in a continuous-time diffusion model by He and Pearson [43] and by Karatzas, Lehoczky, Shreve and Xu [47]. The central idea here is to solve a dual variational problem and then to find the solution of the original problem by convex duality, similarly to the case of a complete model. This powerful approach is applicable to general market models and yields elegant results for the value function and optimal consumption and terminal wealth. However, the optimal portfolio must be then characterized via martingale representation results for the optimal wealth process, so little can be said about the structure and properties of the optimal investments. Another weakness of this approach is the lack of appropriate numerical methods.

The lack of rigorous results for the value function when the utility function is general limits our understanding of the optimal policies. Informally speaking, the optimal portfolio consists of two components. The first is the so-called myopic portfolio and has the same functional form as the one in the classical Merton problem. The second component, usually referred to as the excess hedging demand, is generated by the stochastic factor. Conceptually, very little is understood about this term. In addition, the sum of the two components may become zero which implies that it is optimal for a risk averse investor not to invest in a risky asset with positive risk premium. A satisfactory explanation for this counter-intuitive phenomenon, related to the so-called market participation puzzle (see [5, 18, 44]), is also lacking.

Besides these difficulties, there are other issues that limit the development of an optimal investment theory in complex market environments. One of them is the *static* choice of the utility function at the specific investment horizon. Direct consequences of this choice are, from one hand, the lack of flexibility to revise the risk preferences at other times and, from the other, the inability to assess the performance of investment strategies beyond the pre-specified horizon. Addressing these limitations has been the subject of a number of studies and various approaches have been proposed. With regards to the horizon length, the most popular alternative has been the formulation of the investment problem in $(0, +\infty)$ and incorporating either intermediate consumption or optimizing the investors long-term optimal behavior (see, among others, [48, 81]). Investment models with random horizon have also been examined

([22]). The revision of risk preferences has been partially addressed by recursive utilities (see, for example, [32, 76, 77]). Recently, Musiela and Zariphopoulou developed a forward performance criterion which addresses both issues of the horizon length and revision of risk preferences (see [68, 87]).

Let us now focus on the main subject of this project, the use of stochastic factor models in continuous-time portfolio choice problems, and briefly discuss the existing body of work.

Stochastic factors have been used in portfolio choice to model asset predictability, stochastic volatility, and interest rates. The predictability of stock returns was first discussed in [34, 35, 38]; see also [9, 10, 14, 15]. More complex models were analyzed in [1, 8]. The role of stochastic volatility in investment decisions was studied in [3, 21, 38, 39, 42, 70, 78], and others. Models that combine predictability and stochastic volatility were analyzed, among others, in [50, 54, 61, 86, 71]. In a different modeling direction, stochastic factors have been incorporated in asset allocation models with stochastic interest rates (see, for example, [11, 12, 16, 23, 26, 28, 30, 75, 79, 83]). From the technical point of view, the analysis is not much different. However, various technically interesting questions arise (see, for example, [52, 54, 73]). Classical textbooks on the subject include [17, 31, 48], among others.

More specifically, Korn and Kraft ([52]), Zariphopoulou [86], Pham [71], and Fleming and Hernández-Hernández [36] used stochastic control approach to handle the optimal consumption and asset allocation problems with stochastic opportunity set. However they took some restrictive assumptions on stochastic factors as well as the market price of risk, which excludes models such as Heston's stochastic volatility model. Kramkov and Schachermayer ([55, 56]) provided minimal conditions for the validity of martingale approach on a general semimartingale market. Korn and Kraft ([53]) provided some counter examples to highlight the fact that uncritical application of the two main approaches of solving continuous-time portfolio problems can lead to wrong conclusions if only the necessary and not the sufficient conditions of the main results are checked. Chacko and Viceira [21] considered recursive utility (including power utility) over intermediate consumption. They assumed a stochastic volatility model where the reciprocal of volatility follows a mean-reverting square-root process which is instantly correlated with stock returns. They derived analytic expressions for the optimal consumption and portfolio policies which where exact for the case of power utility and approximate for the case of recursive utility. Castaneda and Hernandez ([20]) used a combination of martingale approach and stochastic control theory to find explicit solutions for power and logarithmic utility functions. Kraft ([54]) proved a verification result which covers Heston's stochastic volatility model for the power utility. Ekeland and Taflin ([33]) studied the problem of optimal portfolio choice in a bond market described in the general HJM framework. They proved the existence of an optimal portfolio in two cases: when the driving Wiener process is finite-dimensional and when the Wiener process is infinite dimensional but the market price of risk is deterministic. Ringer and Tehranchi ([73]) considered the same problem, but with a Markovian Heath–Jarrow–Morton model of the interest rate term structure driven by an infinite-dimensional Wiener process. They gave sufficient conditions for the existence and uniqueness of an optimal trading strategy. Karatzas and Kardaras ([45]) introduced the concept of *numeraire portfolio*, a trading strategy whose wealth appears *better* when compared to the wealth generated by any other strategy, in the sense that the ratio of the two processes is a supermartingale. They derived necessary and sufficient conditions for the numeraire portfolio to exist. Liu ([61]) solved dynamic portfolio choice problems, up to the solution of an ordinary differential equation (ODE), when the asset returns are quadratic and the agent has a power utility function. He also considered three special cases of his model: a pure bond portfolio problem where the bond returns is described by quadratic term structure model (which includes Gaussian and CIR models), a pure stock portfolio problem when the volatility follows Heston model, and a mixed bond-stock portfolio problem with quadratic term structure and Heston stochastic volatility.

The main contribution of this project is developing a new approach to the continuous-time optimal consumption and terminal wealth problem with stochastic opportunity set. It will be called the *direct approach*. It is based on probabilistic arguments, and it can potentially be extended to the general semimartingale market, though we have not tried to do so. Hence, unlike the stochastic control approach, it is not based on the Markov state assumption and it can be used to obtain general results. Its advantage over the martingale approach is that the direct approach, as its name suggests, is dealing with the primal problem *directly*. So, unlike the martingale approach, the completeness or incompleteness of the market has not so much affect on it. Furthermore we are able to obtain the general form of the optimal portfolio policy directly, and not through the dual problem.

The rest of the report is organized as follows. In chapter 2 we defined the market specifications, derived some elementary results, and formalized the portfolio choice problem. Chapter 3 is a literature review about stochastic control approach to portfolio choice. In chapter 4 we present, axiomatically, the martingale approach for incomplete markets. Chapter 5, on generalities of the direct approach, contains the main results of the project. In chapter 6, we considered three special cases: the case of logarithmic utility under general specification of the market, the case of power utility under Black-Scholes type market (the original Merton setting), and the case of power utility with Gaussian term structure. Finally in chapter 7 we have summarized the main results and suggested some topics for further research. For the sake of completeness, we have included, in the appendices, some results on stochastic exponentials and logarithms, change of measure, and Gaussian term structure models.

Chapter 2

Preliminaries

2.1 The market model

We start by defining the market. This definition will be used throughout chapter 4, chapter 5 and section 6.1. But in chapter 3 and section 6.2, we will take some special cases of the definition which will be defined separately. Note that the market is allowed to be *incomplete*, the coefficients are all *stochastic* and *adapted* (but *not necessarily Markov* processes), and the price processes are *without jump.*

Definition 2.1. (General market model) Consider a stochastic basis $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$, where $\mathcal{F}_t$ is the filtration generated by k independent Brownian motions $\boldsymbol{W} = \left(W_t^{(1)}, \ldots, W_t^{(k)}\right)^T$. The market consists of a bank account process $B = (B_t)$, and n other zero-dividend assets (e.g. bonds, stock, etc.) represented by the price process $\boldsymbol{A} = \left(A_t^{(1)}, \ldots, A_t^{(n)}\right)^T$. We always take $n \leq k$. Assume that the price processes are positive continuous semi-martingales (Itô processes) with the following dynamics:

$$\begin{aligned} \boldsymbol{A}_t &= \boldsymbol{A}_0 \mathcal{E}\left(\int_0^\cdot \boldsymbol{\mu}_s ds + \int_0^\cdot \Sigma_s d\boldsymbol{W}_s\right)_t, \\ B_t &= e^{\int_0^t r_s ds}. \end{aligned} \tag{2.1}$$

Here r, the short rate, is an adapted and integrable process. $\boldsymbol{\mu}$, the drift term in real measure, is an $n \times 1$ adapted and integrable vector process. Finally, for all $t \in [0, T]$, Σ_t is an $n \times k$ adapted and almost surely of rank n for all $t \in [0, T]$, and $\Sigma_t \Sigma_t^\top$is integrable.

Remark 2.2. Note that if $n = k$, then the market is complete. In this case Σ_t^{-1} exists for all $t \in [0, T]$.

In the next theorem we parametrize all equivalent martingale measures (EMMs) and state price densities (SPDs) of the market.

Theorem 2.3. *(Parametrization of EMMs and SPDs) Consider the market of definition 2.1. For any EMM $\mathbb{Q}$, there exist a unique predictable process $\boldsymbol{\lambda} = \left(\lambda_t^{(1)}, \ldots, \lambda_t^{(k)}\right)^T$ satisfying the system*

$$\Sigma \boldsymbol{\lambda} = \boldsymbol{\mu} - r\mathbf{1}_{n\times 1}, \tag{2.2}$$

such that $\mathbb{Q}$ *can be written as*

$$\begin{array}{c} \frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T^{(\boldsymbol{\lambda})}, \\ Z^{(\boldsymbol{\lambda})} = \mathcal{E}\left(-\int_0^{\cdot} \boldsymbol{\lambda}_s^{\top} d\boldsymbol{W}_s\right). \end{array} \tag{2.3}$$

Furthermore the corresponding SPD is given by

$$\pi_t = e^{-\int_0^t r_s ds} \mathcal{E}\left(-\int_0^{\cdot} \boldsymbol{\lambda}_s^{\top} d\boldsymbol{W}_s\right)_t, \tag{2.4}$$

and the process $\widetilde{\boldsymbol{W}}$ defined by

$$\widetilde{\boldsymbol{W}}_t \;\triangleq\; \boldsymbol{W}_t + \int_0^t \boldsymbol{\lambda}_s ds, \tag{2.5}$$

is a $\mathbb{Q}$*-Brownian motion.*

Proof. Any measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, can be written as:

$$\begin{array}{c} \frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T^{(\boldsymbol{\lambda})} \\ Z^{(\boldsymbol{\lambda})} = \mathcal{E}\left(-\int_0^{\cdot} \boldsymbol{\lambda}_s^{\top} d\boldsymbol{W}_s\right), \end{array} \tag{2.6}$$

for some predictable process $\boldsymbol{\lambda} = \left(\lambda_t^{(1)}, \ldots, \lambda_t^{(k)}\right)^{\top}$. To see this we may start by defining the martingale $Z_t \triangleq \mathbb{E}_t\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right]$. Then by the representation property of $\boldsymbol{W}$, the stochastic logarithm of Z_t can be represented by $\mathcal{L}\text{og}\,(Z) = -\int_0^{\cdot} \boldsymbol{\lambda}_s^{\top} d\boldsymbol{W}_s$ for some predictable process $\boldsymbol{\lambda}$.

Equation (2.4) follows from the definition of SPD (i.e. $\pi \triangleq \frac{Z^{(\boldsymbol{\lambda})}}{B}$). Theorem B.2 in the appendix implies that $\widetilde{\boldsymbol{W}}$ defined by equation (2.5) is a $\mathbb{Q}$-martingale. Since $\left[\widetilde{\boldsymbol{W}}\right] = t$, it follows that $\widetilde{\boldsymbol{W}}$ is in fact a $\mathbb{Q}$-Brownian motion (this result can also be obtained by the Girsanov theorem).

Consider the discounted price processes $\widetilde{\boldsymbol{A}} \triangleq \frac{\boldsymbol{A}}{B}$, and note that $\mathbb{Q}$ is an EMM if the process $\widetilde{\boldsymbol{A}}$ is a $\mathbb{Q}$-martingale. The $\mathbb{Q}$-dynamics of $\widetilde{\boldsymbol{A}}$ can be found as follow:

$$\begin{aligned} \widetilde{\boldsymbol{A}} &= \frac{1}{B}\boldsymbol{A} \\ &= e^{-\int_0^{\cdot} r_u du} \mathcal{E}\left(\int_0^{\cdot} \boldsymbol{\mu}_u du + \int_0^{\cdot} \Sigma_u d\boldsymbol{W}_u\right) \\ &= \mathcal{E}\left(\int_0^{\cdot} (\boldsymbol{\mu}_u - r_u \mathbf{1}_{n\times 1})\, du + \int_0^{\cdot} \Sigma_u d\boldsymbol{W}_u\right) \\ &= \mathcal{E}\left(\int_0^{\cdot} (\boldsymbol{\mu}_u - r_u \mathbf{1}_{n\times 1})\, du + \int_0^{\cdot} \Sigma_u \left(d\widetilde{\boldsymbol{W}}_u - \boldsymbol{\lambda}_u du\right)\right) \\ &= \mathcal{E}\left(\int_0^{\cdot} (\boldsymbol{\mu}_u - r_u \mathbf{1}_{n\times 1} - \Sigma_u \boldsymbol{\lambda})\, du + \int_0^{\cdot} \Sigma_u d\widetilde{\boldsymbol{W}}_u\right). \end{aligned} \tag{2.7}$$

It follows that $\widetilde{\boldsymbol{A}}$ is a $\mathbb{Q}$-martingale only if equation (2.2) holds. □

Note that if $n < k$ (incomplete market case), then we may use equation (2.2) to express n elements of $\boldsymbol{\lambda}$ in terms of $(k-n)$ other elements of $\boldsymbol{\lambda}$, and in this way we find a parametrization of EMMs in terms of $(k-n)$ *market price of risk* processes. On the other hand, if $n = k$ (the complete market), then equation(2.2) gives us the unique market price of risk process as:

$$\begin{aligned} \boldsymbol{\lambda}_t &= \Sigma_t^{-1} (\boldsymbol{\mu}_t - r_t \mathbf{1}_{n\times 1}) \\ &\triangleq \Sigma_t^{-1} \boldsymbol{b}_t. \end{aligned} \tag{2.8}$$

2.2 Consumption pairs, wealth processes and self-financing strategies

Next, we redefine the well-established concept of self-financing strategies. We will use equation(2.10) of remark 2.5 frequently throughout the chapters 4, 5, and 6. Also note that we used $W^{(\boldsymbol{u},c)}$ for the wealth process, while $\boldsymbol{W}$ is used for the Brownian motion.

Definition 2.4. Take w as the initial wealth. Let $\boldsymbol{u} = \left(u_t^{(1)}, \ldots, u_t^{(n)}\right)^\top_{t\in[0,T]}$ be an adapted process for which the stochastic integrals $\int_0^\cdot \boldsymbol{u}^\top (\operatorname{diag}(\boldsymbol{A}))^{-1} d\boldsymbol{A}$, and $\int_0^\cdot (1 - \boldsymbol{u}^\top \mathbf{1}_{n\times 1}) \frac{dB}{B}$ are well-defined. Also let $c = (c_t)_{t\in[0,T]}$ be a non-negative predictable process. Note that for the process $W^{(\boldsymbol{u},c)} = \left(W_t^{(\boldsymbol{u},c)}\right)_{t\in[0,T]}$, defined by

$$W^{(\boldsymbol{u},c)} \triangleq w\mathcal{E}\left(\int_0^\cdot \boldsymbol{u}^\top (\operatorname{diag}(\boldsymbol{A}))^{-1} d\boldsymbol{A} + \int_0^\cdot (1 - \boldsymbol{u}^\top \mathbf{1}_{n\times 1}) \frac{dB}{B} - \int_0^\cdot c_s ds\right), \tag{2.9}$$

we have $W_t^{(\boldsymbol{u},c)} \geq 0$ for all $t \in [0,T]$. We call $W^{(\boldsymbol{u},c)}$ the wealth process corresponding to a self-financing strategy $\boldsymbol{u}$, the proportional consumption rate process c, and initial wealth w. We also define $SF(w)$ as the set of all pairs $(\boldsymbol{u}, c)$, where $\boldsymbol{u}$ is a self-financing strategy for the consumption rate process c given initial wealth w.

Remark 2.5. From equation (2.1) we have $(\operatorname{diag}(\boldsymbol{A}))^{-1} d\boldsymbol{A} = \boldsymbol{\mu}_u du + \Sigma_u d\boldsymbol{W}_u$. So equation (2.9) can be rewritten as:

$$\begin{aligned} W^{(\boldsymbol{u},c)} &= w\mathcal{E}\left(\int_0^\cdot \boldsymbol{u}_s^\top (\boldsymbol{\mu}_s ds + \Sigma_s d\boldsymbol{W}_s) + \int_0^\cdot (1 - \boldsymbol{u}_s^\top \mathbf{1}_{n\times 1}) r_s ds - \int_0^\cdot c_s ds\right) \\ &= w\mathcal{E}\left(\int_0^\cdot \left\{\boldsymbol{u}_s^\top (\boldsymbol{\mu}_s - r_s \mathbf{1}_{n\times 1}) + r_s - c_s\right\} ds + \int_0^\cdot \boldsymbol{u}_s^\top \Sigma_s d\boldsymbol{W}_s\right). \end{aligned} \tag{2.10}$$

Note that we may re-write equation(2.9) as

$$\begin{cases} \frac{dW_t^{(\boldsymbol{u},c)}}{W_t^{(\boldsymbol{u},c)}} \triangleq \sum_{i=1}^{n} u_t^{(i)} \frac{dA_t^{(i)}}{A_t^{(i)}} + (1 - \sum_{i=1}^{n} u_t^{(i)}) \frac{dB_t}{B_t} - c_t dt, \\ W_0^{(\boldsymbol{u},c)} = w, \end{cases} \tag{2.11}$$

Now we may interpret $u_t^{(i)}$ as the percentage of wealth invested in the i-th asset, $(1 - \sum_{i=1}^{n} u_t^{(i)})$ as the percentage investment in the bank account, and c as the rate of consumption in terms of percentage of wealth.

Also note that we may alternatively consider the total consumption rate C and define, with a little misuse of notation, the wealth process $W^{(\boldsymbol{u},C)} = \left(W_t^{(\boldsymbol{u},C)}\right)_{t\in[0,T]}$ by:

$$\begin{aligned} W_t^{(\boldsymbol{u},C)} &\triangleq w + \int_0^t W_s^{(\boldsymbol{u},C)} \boldsymbol{u}_s^\top \left(\operatorname{diag}\left(\boldsymbol{A}\right)\right)^{-1} d\boldsymbol{A} \\ &\quad + \int_0^t W_s^{(\boldsymbol{u},C)} (1 - \boldsymbol{u}_s^\top \mathbf{1}_{n\times 1}) \frac{dB_s}{B_s} - \int_0^t C_s ds. \end{aligned} \tag{2.12}$$

Then if $W_t^{(\boldsymbol{u},C)} \geq 0$ for all $t \in [0,T]$, we call $\boldsymbol{u}$ a self-financing strategy for the total consumption rate C. Obviously we would have $W^{(\boldsymbol{u},C)} \equiv W^{(\boldsymbol{u},c)}$, if we define

$$c_t \triangleq \frac{C_t}{W_t^{(\boldsymbol{u},C)}}. \tag{2.13}$$

So from now on, we denote the wealth process by $W^{(\boldsymbol{u},c)}$ for both cases of proportional and total consumption. Whether c is a proportional or total consumption will be known from the context.

We will also need the concept of a *consumption pair*, which formalize the consumption behavior of an entity. We identify those *consumption pairs* which can be financed by a self-financing strategy as *affordable consumption* pairs.

Definition 2.6. A *consumption pair* is an ordered pair (C, Z) consisting of an adapted non-negative total consumption rate process $C = (C_t)_{t\in(0,T]}$ with $\int_0^T C_s ds < \infty$, and an $\mathcal{F}_T$-measurable non-negative random variable describing terminal lump-sum consumption Z. We denote the set of all consumption pairs by $\mathcal{C}$.

Definition 2.7. For a consumption pair (C, Z), initial wealth w and a self-financing strategy $\boldsymbol{u}$ (for the total consumption C), we say that $(C, Z, \boldsymbol{u})$ *is budget-feasible at* w if $W_T^{(\boldsymbol{u},C)} \geq Z$ a.s. A consumption pair (C, Z) is *affordable with initial wealth* w, if there exist a self-financing strategy $\boldsymbol{u}$ such that $(C, Z, \boldsymbol{u})$ is budget-feasible at w. We denote by $\mathcal{C}(w)$ the set of all consumption pairs affordable with initial wealth w.

2.3 Problem Formulation

The traditional criterion for optimal portfolio choice has been based on maximal expected utility (for the historical perspective see [88]). The tradition is either assuming an additively time-separable utility function for intermediate consumption of the form $\mathbb{E}\left[\int_0^T u(t, C_t)dt\right]$ (in this case the problem is usually called the *optimal consumption problem*), or utility of terminal wealth only $\mathbb{E}\left[U\left(W_T\right)\right]$ (the portfolio choice problem with this criterion is also called an *asset allocation problem*), or the sum of these two components $\mathbb{E}\left[\int_0^T u(t, C_t)dt + U(W_T)\right]$.

On the other hand, it is worth to mention that it has long been recognized by economists that preferences may not be Intertemporally separable. In particular, the utility associated with the choice of consumption at a given date is likely to depend on past choices of consumption. According to Browning [13], this idea dates back to the 1890 book 'Principles of Economics' by Alfred Marshall. For example high past consumption generates a desire for high current consumption. Generalizations of standard time-separable preferences that have been suggested include *recursive* or *stochastic differential utility* (see, for example, [32, 76, 77]), *habit formation criterion* (see [67]), and *forward performance criterion* (see [68, 87]).

In this project we adapt the traditional criterion of expected utility of terminal wealth and time-separable utility of intermediate consumption. Specifically we define *utility functionals* as follows.

Definition 2.8. A total utility functional $J : \mathcal{C} \to \mathbb{R}$ is defined by:

$$J(C, Z) \triangleq \mathbb{E}\left[\int_0^T u(t, C_t)dt + U(Z)\right], \tag{2.14}$$

with the following assumption on functions $u(.,.)$ and $U(.)$:

- $u : [0, T] \times \mathbb{R}^+ \to \mathbb{R}$ is continuous, and for each $t \in [0, T]$, $u(t, .) : \mathbb{R}^+ \to \mathbb{R}$ is increasing and concave.

- $U : \mathbb{R}^+ \to \mathbb{R}$ is increasing and concave.

- At least one of $u(.,.)$ or $U(.)$ is non-zero. Furthermore, either U is strictly concave or zero, or for all $t \in [0, T]$, $u(., t)$ is strictly concave or zero.

We refer to the function $u\,(.,.)$ as the *consumption utility function*, and the function $U\,(.)$ as the *terminal utility functions*.

We may now give the formal definition of Merton's problem.

Problem 2.9. (Merton's Problem) Consider a total utility functional

$$J(C, Z) \triangleq \mathbb{E}\left[\int_0^T u(t, C_t)dt + U(Z)\right]. \tag{2.15}$$

Then for the initial wealth w, we define Merton's problem as:

$$\sup_{(\boldsymbol{u},c)\in SF(w)} J(cW^{(\boldsymbol{u},c)}, W_T^{(\boldsymbol{u},c)}) \ . \tag{2.16}$$

Chapter 3

Stochastic Control Theory*

In this chapter we present the stochastic control approach without going into details. For the sake of simplicity, we only consider two assets throughout this chapter: a risky stock and a bank account.

As already pointed out in the first chapter, the underlying assumption of the stochastic control approach is assuming a Markov state processes. More specifically we should restrict our definition of market as follows. The stock price process $S = (S_t)_{t\geq 0}$ and the bank account $B = (B_t)_{t\geq 0}$ have the following dynamics:

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu\left(Y_t\right)dt + \sigma\left(Y_t\right)dW_t^{(1)}, \\ \frac{dB_t}{B_t} &= r\left(Y_t\right)dt. \end{aligned} \tag{3.1}$$

The stochastic factor $Y = (Y_t)_{t\geq 0}$ is assumed to satisfy:

$$dY_t = b\left(Y_t\right)dt + d\left(Y_t\right)\left(\rho dW_t^{(1)} + \sqrt{1-\rho^2}dW_t^{(2)}\right). \tag{3.2}$$

Here $\boldsymbol{W}_t = \left(W_t^{(1)}, W_t^{(2)}\right)$ is a standard Brownian motion.

Next we define the *value function* $V\left(w, y, t; T\right)$ as

$$V\left(w, y, t; T\right) = \sup_{(u,c)\in SF(w)} \mathbb{E}\left[\int_t^T u(s, c_s W_s^{(u,c)})ds + U(W_T^{(u,c)})|W_t^{(u,c)} = w, Y_t = y\right]. \tag{3.3}$$

As solution of a stochastic optimization problem, the value function is expected to satisfy the Dynamic Programming Principle (DPP), namely for all $s \in (t, T)$,

$$V\left(w, y, t; T\right) = \sup_{(u,c)\in SF(w)} \mathbb{E}\left[V\left(W_s^{(u,c)}, Y_s, s; T\right)|W_t^{(u,c)} = w, Y_t = y\right]. \tag{3.4}$$

This is a fundamental result in optimal control and has been proved for a wide class of optimization problems. For a detailed discussion on the validity and strongest forms of the DPP in problems with controlled diffusions, we refer the reader to [37]. Key issues are the measurability and continuity of the value function process as well as the compactness of the set of admissible controls. It is worth mentioning that a proof specific to the problem at hand has not been

*This chapter is mainly excerpted from Zariphopoulou [87].

produced to date. Recently, a weak version of the DPP was proposed by Bouchard and Touzi [7] where conditions related to measurable selection and boundness of controls are relaxed.

Besides its technical challenges, the DPP exhibits two important properties of the value function process. Specifically, $V(w, y, s; T)$, $s \in [t, T]$, is a supermartingale for an arbitrary investment strategy and becomes a martingale at an optimum (provided certain integrability conditions hold). One may, then, view $V(w, y, s; T)$ as the intermediate (indirect) utility in the relevant market environment. It is worth noticing, however, that the notions of utility and risk aversion for times $t \in [0, T)$ are tightly connected to the investment opportunities the investor has in the specific market. Observe that the DPP yields a backward in time algorithm for the computation of the maximal utility, starting at expiration with $U\left(W_T^{(u,c)}\right)$ and using the martingale property to compute the solution for earlier times. For this reason, we refer to this formulation of the optimal portfolio choice problem as backward.

Fundamental results in the theory of controlled diffusions yield that if the value function is smooth enough then it satisfies the HJB equation,

$$V_t + bV_y + \frac{1}{2}d^2V_{yy} + \sup_{(u,C)\in SF(w)} \left\{\frac{1}{2}w^2u^2\sigma^2V_{ww} + \{w[(\mu - r)u + r] - C\}V_w + wu\rho\sigma dV_{wy} + u(t, C)\right\} = 0. \tag{3.5}$$

Moreover, optimal policies may be constructed in a feedback form from the first-order conditions in the HJB equation, provided that the candidate feed-back process is admissible and the wealth SDE has a strong solution when the candidate control is used. The latter usually requires further regularity on the value function. In the reverse direction, a smooth solution of the HJB equation that satisfies the appropriate terminal and boundary (or growth) conditions may be identified with the value function, provided the solution is unique in the appropriate sense. These results are usually known as the *verification theorem* and we refer the reader to [37, 85] for a general exposition on the subject.

In maximal expected utility problems, it is rarely the case that the arguments in either direction of the verification theorem can be established. Indeed, it is very difficult to show a priori regularity of the value function, with the main difficulties coming from the lack of global Lipschitz regularity of the coefficients of the controlled process with respect to the controls and from the non-compactness of the set of admissible policies. It is, also, very difficult to establish existence, uniqueness and regularity of the solutions to the HJB equation. This is caused primarily by the presence of the control policy in the volatility of the controlled wealth process which makes the classical assumptions of global Lipschitz conditions of the equation with regards to the non linearities fail. Additional difficulties come from state constraints and the non-compactness of the admissible set. Regularity results for the value function (3.4) for general utility functions have not been obtained to date except for the special cases of homothetic preferences (see, for example, [36, 54, 66, 71, 86]). The most general result in this direction, and in a much more general market model, was recently obtained by Kramkov and Sîrbu [57] where it is shown that the value function is twice differentiable in the spatial argument but without establishing its continuity.

Some answers to the questions related to the characterization of the solutions to the HJB equation may be given if one relaxes the requirement to have classical solutions. An appropriate class of weak solutions turns out to be the so called viscosity solutions ([27, 60, 59], and [82]). The analysis and characterization of the value function in the viscosity sense has been carried out for the special cases of power and exponential utility (see, for example, [86]). However, proving that the value function is the unique viscosity solution of (3.5) has not been addressed.

A key property of viscosity solutions is their robustness (see [59]). If the HJB has a unique viscosity solution (in the appropriate class), robustness can be used to establish convergence of numerical schemes for the value function and the optimal feedback laws. Such numerical studies have been carried out successfully for a number of applications. However, for the model at hand, no such studies are available. Numerical results using Monte Carlo techniques have been obtained in [29] for a model more general than the one herein.

Important questions arise on the dependence, sensitivity and robustness of the optimal feedback portfolio in terms of the market parameters, the wealth, the level of the stochastic factor and the risk preferences. Such questions are central in financial economics and have been studied, primarily in simpler models in which intermediate consumption is also incorporated (see, among others, [2, 51, 58, 69], and [74]). For diffusion models with and without a stochastic factor, qualitative results can be found in [29, 50, 61, 84], and, recently, in [6] (see, also [62] for a general incomplete market discrete model). However, a qualitative study for general utility functions and/or arbitrary factor dynamics has not been carried out to date.

Let's have a review on the existing results for the most frequently used utilities, namely, the exponential, power and logarithmic ones. They have convenient homogeneity properties which, in combination with the linearity of the wealth dynamics in the control policies, enable us to reduce the HJB equation to a quasilinear one. Under a *distortion transformation* (see, for example, [86]) the latter can be linearized and solutions in closed form can be produced using the Feynman-Kac formula. The smoothness of the value function and, in turn, the verification of the optimal feedback policies follows easily. Multi-factor models for these preferences have been analyzed by various authors. The theory of BSDE has been successfully used to characterize and represent the solutions of the reduced HJB equation (see [49]). The regularity of its solutions has been studied using PDE arguments by [71] and [66], for power and exponential utilities, respectively. Finally, explicit solutions for a three factor model can be found in [61].

Chapter 4

Martingale or Duality Approach

The Martingale method has been given increasing attention since it was conducted by Pliska [72], Cox and Huang [24, 25] and Karatzas, Lehoczky and Shreve [46]. Martingale method allows us to solve the problems of utility maximization in a very elegant manner. However, the Martingale method is not omnipotent. When the market is incomplete, traditional Martingale method will be problematic. He and Pearson [43], and Karatzas, Lehoczky, Shreve and Xu [47] generalized the approach to incomplete markets. The central idea here is to solve a dual variational problem and then to find the solution of the original problem by convex duality.

In a general incomplete market, the dual problem is a stochastic variational problem where the control variables are the consumption process c, the terminal wealth Z, and market price of risk process$\boldsymbol{\lambda}$. Namely,

$$\begin{array}{l} \sup \quad J(C,Z) \triangleq \mathbb{E}\left[\int_0^T u(t,C_t)dt + U(Z)\right] \\ \text{subject to :} \\ \qquad \mathbb{E}\left[\int_0^T \pi_t^{(\boldsymbol{\lambda})} C_t dt + \pi_T^{(\boldsymbol{\lambda})} Z\right] = w, \\ \boldsymbol{\lambda} = \Sigma \left(\Sigma\Sigma^T\right)^{-1} \boldsymbol{b} + \left\{\Sigma^\top \left(\Sigma\Sigma^\top\right)^{-1} \Sigma - I_{k\times k}\right\} \boldsymbol{a}^{(C,Z,\boldsymbol{\lambda})}, \\ (C,Z) \in \mathcal{C},\ \boldsymbol{\lambda} \text{ is an MPR.} \end{array} \tag{4.1}$$

see remark 4.8 below. For general semimartingale markets and general utility functional, necessary and sufficient conditions for validity of the martingale approach has been proposed, see [55, 56]. These conditions guarantee that the duality gap is zero (i.e. the optimal value of the primal and dual problem are the same), and that the optimal consumption pair (c, Z) given by the dual problem is actually affordable. However they only provide the existence of the optimal portfolio $\boldsymbol{u}$, and to find the portfolio policy one should solve a representation problem.

The situation is much more tractable once we assume that the market is complete. In this case the market price of risk is unique, hence the dual problem would be only in terms of the consumption pair (c, Z). But then we can solve this *static* problem directly to find the optimal consumption and terminal wealth. Specifically the optimal consumption pair is,

$$\begin{array}{l} C_t^* \triangleq I(\eta\pi_t, t) \text{ for } t \in (0,T)\,, \\ Z^* \triangleq I_F(\eta\pi_T). \end{array} \tag{4.2}$$

where $I_F(.)$ and $I(., t)$ are the inverses of $U'(x)$ and $\frac{\partial}{\partial x}u(x,t)$, $\pi = (\pi_t)$ is the state price density process, and the constant η is the unique solution of the following equation:

$$\mathbb{E}\left[\pi_T I_F(\eta\pi_T) + \int_0^T \pi_\tau I(\eta\pi_\tau, \tau)d\tau\right] = w. \tag{4.3}$$

See remark 4.10 below. This is actually the original martingale approach. Yet again, to obtain the optimal portfolio we need to solve a representation problem.

In the next subsection we present the Martingale approach in details.

4.1 Proof of the Martingale Approach

The essence of the martingale approach in incomplete markets, also known as duality approach, is in remark 4.8 and theorem 4.9. In summary the proof goes like this:

- First we identify a necessary and sufficient condition under which an arbitrary consumption pair (C, Z) is affordable. It will be done in two steps, lemma 4.6 and proposition 4.7.
- Then we relate Merton's problem to the dual problem containing only the consumption pairs and not the optimal portfolio $\boldsymbol{u}$ (see equation 4.26).
- Then in theorem 4.9, we identify conditions under which we can solve this dual problem.
- Finally we consider the case of complete markets in remark4.10.

To express the martingale approach, we need the following definitions and lemmas.

Definition 4.1. Assume that we are given a consumption pair (C, Z) and a market-price of risk process $\boldsymbol{\lambda}$ (along with the corresponding SPD $\pi^{(\boldsymbol{\lambda})}$). Then we define the process $W^{(C,Z,\boldsymbol{\lambda})}$ as:

$$\pi_t^{(\boldsymbol{\lambda})} W_t^{(C,Z,\boldsymbol{\lambda})} \triangleq \mathbb{E}_t\left[\pi_T^{(\boldsymbol{\lambda})} Z + \int_t^T \pi_\tau^{(\boldsymbol{\lambda})} C_\tau d\tau\right]. \tag{4.4}$$

Lemma 4.2. (Inada condition) *We say that a strictly concave, increasing and differentiable function $F : \mathbb{R}^+ \to \mathbb{R}$ satisfies Inada conditions if* $\inf_x F'(x) = 0$ *and* $\sup_x F' = +\infty$. *If $F : \mathbb{R}^+ \to \mathbb{R}$ satisfies Inada conditions, then the function I_F, which is the inverse of F', is well defined as a strictly decreasing continuous function on* $(0, \infty)$ *which image is* $(0, \infty)$.

Definition 4.3. *(Condition A)* Either $U(x)$ is zero or $U(x)$ is differentiable on $(0, \infty)$, strictly concave and satisfies Inada conditions. Either (for all t) $u(x, t)$ is zero or (for all t) $u(x, t)$ is differentiable on $(0, \infty)$, strictly concave and satisfies Inada conditions. Also for all $\eta > 0$ and any SPD π we assume:

$$\mathbb{E}\left[\pi_T I_F(\eta\pi_T) + \int_0^T \pi_\tau I(\eta\pi_\tau, \tau)d\tau\right] < \infty. \tag{4.5}$$

Here $I_F(.)$ and $I(., t)$ are the inverses of $U'(x)$ and $\frac{\partial}{\partial x}u(x, t)$.

Lemma 4.4. *Consider problem 2.9. Assume that* condition A *holds, then for any MPR* $\boldsymbol{\lambda}$*, there exists a unique constant* $\eta(\boldsymbol{\lambda}) > 0$ *such that:*

$$\mathbb{E}\left[\pi_T^{(\boldsymbol{\lambda})} I_F(\eta(\boldsymbol{\lambda})\,\pi_T^{(\boldsymbol{\lambda})}) + \int_0^T \pi_\tau^{(\boldsymbol{\lambda})} I(\eta(\boldsymbol{\lambda})\,\pi_\tau^{(\boldsymbol{\lambda})}, \tau)d\tau\right] = w. \tag{4.6}$$

Here $I(.,t)$ *and* $I_F(.)$ *are the inverses of the derivatives of* $u(.,t)$ *and* $U(.)$*, respectively.*

Proof. Define

$$f(\eta) \triangleq \mathbb{E}\left[\pi_T^{(\boldsymbol{\lambda})} I_F(\eta\pi_T^{(\boldsymbol{\lambda})}) + \int_0^T \pi_\tau^{(\boldsymbol{\lambda})} I(\eta\pi_\tau^{(\boldsymbol{\lambda})}, \tau)d\tau\right]. \tag{4.7}$$

Condition A along with lemma 4.2 imply that one or both of $I_F(.)$ and $I(.,t)$ (for all $t \in (0,T)$) are strictly decreasing continuous functions on $(0,\infty)$ which images are $(0,\infty)$. Since $f(.)$ inherits these two properties, we can conclude that for any $w > 0$ there exist a unique $\eta > 0$ which satisfies $f(\eta) = w$. We denote this unique solution by $\eta(\boldsymbol{\lambda})$. □

We need to prove the numeraire invariance property of our definition of budget feasibility (definition 2.7).

Lemma 4.5. *(numeraire invariance) Let*$\pi = (\pi_t)$ *be an arbitrary deflator and* w *be the initial wealth. A strategy* $(C, Z, \boldsymbol{u})$ *is budget feasible at* w *if and only if :*

$$\begin{cases} \pi_t W_t^{(\boldsymbol{u},C)} = \pi_0 w + \int_0^t \pi_s W_s^{(\boldsymbol{u},C)} \boldsymbol{u}_s^\top \left(\operatorname{diag}(\pi_s \boldsymbol{A}_s)\right)^{-1} d(\pi\boldsymbol{A})_s & t \in [0,T] \\ \qquad + \int_0^t \pi_s W_s^{(\boldsymbol{u},C)} \left(1 - \boldsymbol{u}_s^\top \mathbf{1}_{n\times 1}\right) \frac{d(\pi B)_s}{\pi_s B_s} - \int_0^t \pi_s C_s ds \geq 0 & \\ \pi_T W_T^{(\boldsymbol{u},C)} \geq \pi_T Z \quad a.s. & \end{cases} \tag{4.8}$$

Proof. The proof is a simple application of product rule. By definition (C, Z, u) is budget feasible if and only if:

$$\begin{cases} W_t^{(\boldsymbol{u},C)} = w + \int_0^t W_s^{(\boldsymbol{u},C)} \boldsymbol{u}_s^\top \left(\operatorname{diag}(\boldsymbol{A}_s)\right)^{-1} d\boldsymbol{A}_s & t \in [0,T] \\ \qquad + \int_0^t W_s^{(\boldsymbol{u},C)} \left(1 - \boldsymbol{u}_s^\top \mathbf{1}_{n\times 1}\right) \frac{dB}{B} - \int_0^t C_s ds \geq 0 & \\ W_T^{(\boldsymbol{u},C)} \geq Z \quad a.s. & \end{cases} \tag{4.9}$$

The second conditions in equation (4.8) are equivalent to the second equations above.
To show that the first condition in equation(4.8) is implied by the first condition in equation(4.9), we apply the product rule to obtain:

$$d\left(\pi_t W_t^{(\boldsymbol{u},C)}\right) = \pi_t dW_t^{(\boldsymbol{u},C)} + W_t^{(\boldsymbol{u},C)} d\pi_t + d\left[\pi, W^{(\boldsymbol{u},C)}\right]_t \tag{4.10}$$

Note that:

$$\begin{aligned} d\left[\pi, W^{(\boldsymbol{u},C)}\right]_t &= d\left[\pi, \int_0^\cdot W_s^{(\boldsymbol{u},C)} \boldsymbol{u}_s^\top \left(\operatorname{diag}(\boldsymbol{A}_s)\right)^{-1} d\boldsymbol{A}_s\right] \\ &= W_t^{(\boldsymbol{u},C)} \boldsymbol{u}_t^\top \left(\operatorname{diag}(\boldsymbol{A}_t)\right)^{-1} d[\pi, \boldsymbol{A}]_t \end{aligned}$$

By substituting for $d\left[\pi, W^{(\boldsymbol{u},C)}\right]$ and $dW_t^{(\boldsymbol{u},C)}$ in equation(4.10), we will obtain:

$$\begin{aligned} d\left(\pi W^{(u)}\right)_t &= \pi_t \left\{ W_t^{(\boldsymbol{u},C)} \boldsymbol{u}_t^\top \left(\text{diag}\left(\boldsymbol{A}_t\right)\right)^{-1} d\boldsymbol{A}_t + W_t^{(\boldsymbol{u},C)} \left(1 - \boldsymbol{u}_t^\top \mathbf{1}_{n\times 1}\right) \frac{dB_t}{B_t} - C_t dt \right\} \\ &\quad + W_t^{(\boldsymbol{u},C)} d\pi_t + W_t^{(\boldsymbol{u},C)} \boldsymbol{u}_t^\top \left(\text{diag}\left(\boldsymbol{A}_t\right)\right)^{-1} d[\pi, \boldsymbol{A}]_t \\ &= W_t^{(\boldsymbol{u},C)} \boldsymbol{u}_t^\top \left(\text{diag}\left(\boldsymbol{A}_t\right)\right)^{-1} \left\{\pi_t d\boldsymbol{A}_t + \text{diag}\left(\boldsymbol{A}_t\right) \mathbf{1}_{n\times 1} d\pi_t + d[\pi, \boldsymbol{A}]_t\right\} \\ &\quad + W_t^{(\boldsymbol{u},C)} \left(1 - \boldsymbol{u}_t^\top \mathbf{1}_{n\times 1}\right) \frac{\left\{\pi_t dB_t + B_t d\pi_t\right\}}{B_t} - \pi_t C_t dt \\ &= W_t^{(\boldsymbol{u},C)} \boldsymbol{u}_t^\top \left(\text{diag}\left(\boldsymbol{A}_t\right)\right)^{-1} d\left(\pi \boldsymbol{A}\right)_t + W_t^{(\boldsymbol{u},C)} \left(1 - \boldsymbol{u}_t^\top \mathbf{1}_{n\times 1}\right) \frac{d\left(\pi B\right)_t}{B_t} - \pi_t C_t dt \end{aligned} \tag{4.11}$$

which in turn, will give us the first condition in equation(4.8).

Finally to show the converse, i.e. that the first condition in equation(4.9) is also implied by the first condition in equation(4.8), we start by defining the process as $\boldsymbol{A}_t^{(\pi)} = \pi_t \boldsymbol{A}_t$, $B_t^{(\pi)} = \pi_t B_t, C_t^{(\pi)} = \pi_t C_t$, and $Z^{(\pi)} = \pi_T Z$. Then equation(4.8) is equivalent to assuming that $(C^{(\pi)}, Z^{(\pi)}, u)$ is budget budget feasible at w with price processes given by $\boldsymbol{A}_t^{(\pi)}$ and $B_t^{(\pi)}$. Now by considering the deflator $\frac{1}{\pi_t}$ and using the same argument we used form equation(4.10) to (4.11), we will reach the first condition in equation (4.9). □

As mentioned above, the main idea of martingale approach is to identify conditions under which an arbitrary consumption pair (C, Z) is affordable. The following lemma is the first step towards this goal.

Lemma 4.6. *Assume we are given a consumption pair (C, Z) and a market-price of risk process $\boldsymbol{\lambda}$. Then (C, Z) is affordable with initial wealth w if and only if :*
(i)-

$$\mathbb{E}\left[\int_0^T \pi_t^{(\boldsymbol{\lambda})} C_t dt + \pi_t^{(\boldsymbol{\lambda})} Z\right] \leq w. \tag{4.12}$$

(ii)- There exist a predictable process $\boldsymbol{u} = \left(u_t^{(1)}, \ldots, u_t^{(n)}\right)_{t\in[0,T]}^T$ satisfying:

$$\Sigma_t^T \boldsymbol{u}_t - \boldsymbol{\lambda}_t = \frac{\left[\boldsymbol{W}, \pi^{(\boldsymbol{\lambda})} W^{(C,Z,\boldsymbol{\lambda})}\right]_t'}{\pi_t^{(\boldsymbol{\lambda})} W_t^{(C,Z,\boldsymbol{\lambda})}}. \tag{4.13}$$

Furthermore if these conditions hold, then $\boldsymbol{u}$ is the self-financing strategy for the consumption pair (C, Z).

Proof. First we prove the necessity of condition *(i)*. Assume (C, Z) to be affordable, i.e. there exist a self-financing strategy $\boldsymbol{u}$ such that $(C, Z, \boldsymbol{u})$ is budget feasible at w. By using lemma4.5 for deflator π, we will obtain:

$$\begin{cases} \pi_t W_t^{(\boldsymbol{u},C)} = w + \int_0^t \pi_s W_s^{(\boldsymbol{u},C)} \boldsymbol{u}_s^\top \left(\text{diag}\left(\pi_s \boldsymbol{A}_s\right)\right)^{-1} d\left(\pi \boldsymbol{A}\right)_s & t \in [0, T] \\ \qquad + \int_0^t \pi_s W_s^{(\boldsymbol{u},C)} \left(1 - \boldsymbol{u}_s^\top \mathbf{1}_{n\times 1}\right) \frac{d(\pi B)_s}{\pi_s B_s} - \int_0^t \pi_s C_s ds \geq 0 & \\ \pi_T W_T^{(\boldsymbol{u},C)} \geq \pi_T Z \quad a.s. & \end{cases}$$

Define a process $\mathcal{N}$ by:

$$\begin{aligned}\mathcal{N}_t &\triangleq \pi_t W_t^{(\boldsymbol{u},C)} + \int_0^t \pi_s C_s ds &(4.14)\\ &= w + \int_0^t W_s^{(\boldsymbol{u},C)} \boldsymbol{u}_s^\top \left(\operatorname{diag}\left(\boldsymbol{A}_s\right)\right)^{-1} d\left(\pi \boldsymbol{A}\right)_s + \int_0^t W_s^{(\boldsymbol{u},C)} \left(1 - \boldsymbol{u}_s^\top \mathbf{1}_{n\times 1}\right) \frac{d\left(\pi B\right)_s}{B_s} &(4.15)\end{aligned}$$

Note that by definition, $\left(\pi^{(\lambda)} A\right)$ and $\left(\pi^{(\lambda)} B\right)$ are (local) martingales, so $\mathcal{N}$ is a non-negative local martingale, and hence a super martingale. The supper-martingale property give us the desired equation (4.12):

$$\mathbb{E}\left[\pi_T Z + \int_0^T \pi_t C_t dt\right] \leq \mathbb{E}\left[\mathcal{N}_T\right] \leq \mathcal{N}_0 = w. \tag{4.16}$$

To prove the necessity of condition *(ii)* we may proceed as follow. From remark (2.5), $W^{(C,Z,\boldsymbol{\lambda})}$ is the wealth process of a self-financing strategy $\boldsymbol{u}$ with $c \triangleq \frac{C}{W^{(C,Z,\boldsymbol{\lambda})}}$ as the proportional consumption and w as the initial wealth, if and only if:

$$W_t^{(C,Z,\boldsymbol{\lambda})} = w\mathcal{E}\left(\int_0^\cdot \left\{\boldsymbol{u}_u^\top \left(\boldsymbol{\mu}_u - r_u \mathbf{1}_{2\times 1}\right) + r_u - c_u\right\} du + \int_0^\cdot \boldsymbol{u}_u^\top \Sigma_u d\boldsymbol{W}_u\right). \tag{4.17}$$

Recall from remark (2.3) that $\pi \triangleq \frac{Z^{(\boldsymbol{\lambda})}}{B}$, where $Z^{(\boldsymbol{\lambda})} = \mathcal{E}\left(-\int_0^\cdot \boldsymbol{\lambda}_s^\top d\boldsymbol{W}_s\right)$. So by using the properties of stochastic integrals we may write:

$$\begin{aligned}\pi^{(\boldsymbol{\lambda})} W^{(C,Z,\boldsymbol{\lambda})} &= \frac{Z^{(\lambda)}}{B} W^{(C,Z,\boldsymbol{\lambda})}\\ &= \frac{\mathcal{E}\left(-\int_0^\cdot \boldsymbol{\lambda}_s^\top d\boldsymbol{W}_s\right)}{B_0 e^{\int_0^\cdot r_s ds}}\\ &\quad \times \left\{w\mathcal{E}\left(\int_0^\cdot \left\{\boldsymbol{u}_u^\top \left(\boldsymbol{\mu}_u - r_u \mathbf{1}_{2\times 1}\right) + r_u - c_u\right\} du + \int_0^\cdot \boldsymbol{u}_u^\top \Sigma_u d\boldsymbol{W}_u\right)\right\}\\ &= \frac{w}{B_0} e^{\left[-\int_0^\cdot \boldsymbol{\lambda}_s^T d\boldsymbol{W}_s, \int_0^\cdot \boldsymbol{u}_s^T \Sigma_s d\boldsymbol{W}_s\right]}\\ &\quad \times \mathcal{E}\left(\int_0^\cdot \left\{\boldsymbol{u}_s^\top \left(\boldsymbol{\mu}_s - r_s \mathbf{1}_{n\times 1}\right) - c_s\right\} ds + \int_0^\cdot \left(\boldsymbol{u}_s^\top \Sigma_s - \boldsymbol{\lambda}_s^\top\right) d\boldsymbol{W}_s\right)\\ &= \frac{w}{B_0} \mathcal{E}\left(\int_0^\cdot \left\{\boldsymbol{u}_s^\top \left(\boldsymbol{\mu}_s - r_s \mathbf{1}_{2\times 1} - \Sigma_s \boldsymbol{\lambda}_s\right) - c_s\right\} ds + \int_0^\cdot \left(\boldsymbol{u}_s^\top \Sigma_s - \boldsymbol{\lambda}_s^\top\right) d\boldsymbol{W}_s\right)\\ &= \frac{w}{B_0} \mathcal{E}\left(-\int_0^\cdot c_s ds + \int_0^\cdot \left(\boldsymbol{u}_s^\top \Sigma_s - \boldsymbol{\lambda}_s^\top\right) d\boldsymbol{W}_s\right). &(4.18)\end{aligned}$$

Hence we obtain:

$$\begin{aligned}d\left(\pi^{(\boldsymbol{\lambda})} W^{(C,Z,\boldsymbol{\lambda})}\right)_t &= \pi_t^{(\boldsymbol{\lambda})} W_t^{(C,Z,\boldsymbol{\lambda})} \left(-c_t dt + \left(\boldsymbol{u}_t^\top \Sigma_t - \boldsymbol{\lambda}_t^\top\right) d\boldsymbol{W}_t\right)\\ &= -\pi_t^{(\boldsymbol{\lambda})} C_t dt + \pi_t^{(\boldsymbol{\lambda})} W_t^{(C,Z,\boldsymbol{\lambda})} \left(\boldsymbol{u}_t^\top \Sigma_t - \boldsymbol{\lambda}_t^\top\right) d\boldsymbol{W}_t. &(4.19)\end{aligned}$$

On the other hand from equations (4.4) and the martingale representation property of $\boldsymbol{W}$, we have

$$d\left(\pi^{(\boldsymbol{\lambda})} W^{(C,Z,\boldsymbol{\lambda})}\right)_t + \pi_t^{(\boldsymbol{\lambda})} C_t dt = \left[\pi^{(\boldsymbol{\lambda})} W^{(C,Z,\boldsymbol{\lambda})}, \boldsymbol{W}\right]_t' d\boldsymbol{W}_t. \tag{4.20}$$

From the last two results we can conclude $\pi_t^{(\boldsymbol{\lambda})} W_t^{(C,Z,\boldsymbol{\lambda})} \left(\boldsymbol{u}_t^\top \Sigma_t - \boldsymbol{\lambda}_t^\top\right) = \left[\pi^{(\boldsymbol{\lambda})} W^{(C,Z,\boldsymbol{\lambda})}, \boldsymbol{W}\right]_t'$, which is equation(4.13).

Finally to prove the converse, suppose that conditions *(i)* and *(ii)* hold. Then by defining

$$W^{(\boldsymbol{u},c)} \triangleq w\mathcal{E}\left(\int_0^\cdot \left\{\boldsymbol{u}_u^\top \left(\boldsymbol{\mu}_u - r_u \mathbf{1}_{2\times 1}\right) + r_u - c_u\right\} du + \int_0^\cdot \boldsymbol{u}_u^\top \Sigma_u d\boldsymbol{W}_u\right),$$

we may invert the proof of necessity of condition *(ii)*, to conclude that $W^{(C,Z,\boldsymbol{\lambda})} \equiv W^{(\boldsymbol{u},c)}$. Now we may use the definition of $W^{(C,Z,\boldsymbol{\lambda})}$ along with condition *(i)* to prove that $\boldsymbol{u}$ is indeed a self-financing strategy and $(C, Z, \boldsymbol{u})$ is budget feasible. □

In the following proposition we improve the result of the previous lemma by making the affordability condition free of any self-financing strategy $\boldsymbol{u}$. Also we give an expression for the corresponding self-financing strategy, if a consumption pair is affordable.

Proposition 4.7. *A consumption pair (C, Z) is affordable with initial wealth w if and only if for a specific market price of risk $\boldsymbol{\lambda}$ satisfying*

$$\begin{aligned}\boldsymbol{\lambda} &\triangleq \Sigma^\top \left(\Sigma\Sigma^T\right)^{-1} \left(\boldsymbol{\mu} - r\mathbf{1}_{n\times 1}\right) \\ &+ \left\{\Sigma^\top \left(\Sigma\Sigma^\top\right)^{-1} \Sigma - I_{k\times k}\right\} \left(\frac{\left[\boldsymbol{W}, \pi^{(\boldsymbol{\lambda})} W^{(C,Z,\boldsymbol{\lambda})}\right]_t'}{\pi_t^{(\boldsymbol{\lambda})} W_t^{(C,Z,\boldsymbol{\lambda})}}\right),\end{aligned} \tag{4.21}$$

we have

$$\mathbb{E}\left[\int_0^\top \pi_t^{(\boldsymbol{\lambda})} C_t dt + \pi_t^{(\boldsymbol{\lambda})} Z\right] \leq w. \tag{4.22}$$

Furthermore, if this condition holds, the corresponding self-financing strategy is given by:

$$\boldsymbol{u} = \left(\Sigma\Sigma^\top\right)^{-1} \left\{\boldsymbol{\mu} - r\mathbf{1}_{n\times 1} + \Sigma\left(\frac{\left[\boldsymbol{W}, \pi^{(\boldsymbol{\lambda})} W^{(C,Z,\boldsymbol{\lambda})}\right]_t'}{\pi_t^{(\boldsymbol{\lambda})} W_t^{(C,Z,\boldsymbol{\lambda})}}\right)\right\}. \tag{4.23}$$

Proof. Define $\boldsymbol{a}_t^{(C,Z,\boldsymbol{\lambda})} \triangleq \frac{\left[\boldsymbol{W}, \pi^{(\boldsymbol{\lambda})} W^{(C,Z,\boldsymbol{\lambda})}\right]_t'}{\pi_t^{(\boldsymbol{\lambda})} W_t^{(C,Z,\boldsymbol{\lambda})}}$ and $\boldsymbol{b}_t \triangleq \boldsymbol{\mu} - r\mathbf{1}_{n\times 1}$. Then from condition *(ii)* of lemma (4.6), and theorem (2.3) we have the following system for vectors $\boldsymbol{\lambda}$ and $\boldsymbol{u}$:

$$\begin{cases}\Sigma_t^T \boldsymbol{u}_t - \boldsymbol{\lambda}_t = \boldsymbol{a}_t^{(C,Z,\boldsymbol{\lambda})} \\ \Sigma_t \boldsymbol{\lambda}_t = \boldsymbol{b}_t.\end{cases} \tag{4.24}$$

By "solving" for $\boldsymbol{\lambda}$ from the first equation and substituting in the second equation we will obtain

$$\Sigma_t \left(\Sigma_t^T \boldsymbol{u}_t - \boldsymbol{a}_t^{(C,Z,\boldsymbol{\lambda})}\right) = \boldsymbol{b}_t. \tag{4.25}$$

Note that $\operatorname{rank}\left(\Sigma\Sigma^T\right) = \operatorname{rank}(\Sigma) = n$, so the $n \times n$ matrix $\Sigma\Sigma^T$ is invertible. Now by rearranging the last result we will obtain equation (4.23). By substituting for $\boldsymbol{u}$ in the first equation of the system above, we will obtain equation (4.21). Hence the necessary and sufficient condition of lemma (4.6) can be restated as this proposition. □

Remark 4.8. Note that from proposition (4.7), Merton's problem is equivalent to the following variational problem:

$$\begin{array}{l} \sup \quad J(C,Z) \triangleq \mathbb{E}\left[\int_0^T u(t,C_t)dt + U(Z)\right] \\ \text{subject to :} \\ \qquad \mathbb{E}\left[\int_0^T \pi_t^{(\boldsymbol{\lambda})} C_t dt + \pi_T^{(\boldsymbol{\lambda})} Z\right] = w, \\ \boldsymbol{\lambda} = \Sigma\left(\Sigma\Sigma^T\right)^{-1} \boldsymbol{b} + \left\{\Sigma^T\left(\Sigma\Sigma^T\right)^{-1}\Sigma - I_{k\times k}\right\} \boldsymbol{a}^{(C,Z,\boldsymbol{\lambda})}, \\ (C,Z) \in \mathcal{C},\ \boldsymbol{\lambda} \text{ is an MPR.} \end{array} \tag{4.26}$$

This problem, generally known as the dual of problem , can be defined even in a general semimartingale setting (see [55, 56]).

Now we are ready to prove the main result.

Theorem 4.9. *(Martingale Approach) Consider problem 2.9 and assume that* condition A *holds. Define*

$$\begin{array}{l} C_t^{(\boldsymbol{\lambda})} \triangleq I(\eta(\boldsymbol{\lambda})\,\pi_t^{(\boldsymbol{\lambda})}, t) \text{ for } t \in (0,T), \\ Z^{(\boldsymbol{\lambda})} \triangleq I_F(\eta(\boldsymbol{\lambda})\,\pi_T^{(\boldsymbol{\lambda})}), \end{array} \tag{4.27}$$

where $I_F(.)$ and $I(.,t)$ are the inverses of $U'(x)$ and $\frac{\partial}{\partial x}u(x,t)$. Also define $\boldsymbol{b}_t \triangleq \boldsymbol{\mu} - r\mathbf{1}_{n\times 1}$, and $\boldsymbol{a}_t^{(C,Z,\boldsymbol{\lambda})} \triangleq \frac{\left[\boldsymbol{W}, \pi^{(\boldsymbol{\lambda})} W^{(C,Z,\boldsymbol{\lambda})}\right]_t'}{\pi_t^{(\boldsymbol{\lambda})} W_t^{(C,Z,\boldsymbol{\lambda})}}$ so that

$$d\pi_t^{(\boldsymbol{\lambda})} W_t^{(C,Z,\boldsymbol{\lambda})} = \pi_t^{(\boldsymbol{\lambda})} W_t^{(C,Z,\boldsymbol{\lambda})} \boldsymbol{a}_t^{(C,Z,\boldsymbol{\lambda})} d\boldsymbol{W}_t. \tag{4.28}$$

If an MPR $\boldsymbol{\lambda}$ satisfies

$$\boldsymbol{\lambda} = \Sigma^\top\left(\Sigma\Sigma^\top\right)^{-1} \boldsymbol{b} + \left\{\Sigma^\top\left(\Sigma\Sigma^\top\right)^{-1}\Sigma - I_{k\times k}\right\} \boldsymbol{a}^{(C^{(\boldsymbol{\lambda})},Z^{(\boldsymbol{\lambda})},\boldsymbol{\lambda})}, \tag{4.29}$$

then, under that specific MPR $\boldsymbol{\lambda}$, the optimal consumption is $C_t^{(\boldsymbol{\lambda})}$ and the optimal terminal wealth is $Z^{(\boldsymbol{\lambda})}$. Furthermore the corresponding optimal self-financing strategy is given by:

$$\boldsymbol{u} = \left(\Sigma\Sigma^\top\right)^{-1}\left(\boldsymbol{b} + \Sigma\boldsymbol{a}^{(C^{(\boldsymbol{\lambda})},Z^{(\boldsymbol{\lambda})},\boldsymbol{\lambda})}\right), \tag{4.30}$$

and we have $W^{(\boldsymbol{u},C)} = Z^{(\boldsymbol{\lambda})}$.

Proof. Consider the following problem:

$$\begin{array}{l} \sup \quad J(C,Z) \triangleq \mathbb{E}\left[\int_0^T u(t,C_t)dt + U(Z)\right] \\ \text{subject to} \\ \qquad \mathbb{E}\left[\int_0^T \pi_t^{(\boldsymbol{\lambda})} C_t dt + \pi_T^{(\boldsymbol{\lambda})} Z\right] = w, \\ (C,Z) \in \mathcal{C},\ \boldsymbol{\lambda} \text{ is a } \textit{given} \text{ MPR.} \end{array} \tag{4.31}$$

Since there is not enough constraint to make the consumption pair (C,Z) affordable under $\boldsymbol{\lambda}$, this problem is a relaxation of Merton's problem. This means that the solution of this problem gives an upper bound for the Merton's optimal solution. Now the main idea is that if this solution of this relaxation problem satisfies equation (4.21) as well, then it would be the solution for Merton's problem.

First we try to find the solution of equation (4.31). The Lagrangian is:

$$\begin{aligned}\mathcal{L}(C,Z,\eta) &\triangleq \mathbb{E}\left[\int_0^T u(t,C_t)dt + U(Z)\right] \\ &\quad -\eta\left(\mathbb{E}\left[\int_0^T \pi_t^{(\boldsymbol{\lambda})} C_t dt + \pi_T^{(\boldsymbol{\lambda})} Z\right] - w\right).\end{aligned} \tag{4.32}$$

To check the first order conditions we first find the directional derivatives:

$$\begin{aligned}\tfrac{d}{d\varepsilon}\mathcal{L}(C+\varepsilon x, Z,\eta) &= \mathbb{E}\left[\int_0^T \left(u'(t,C_t+\varepsilon x_t) - \eta\pi_t^{(\boldsymbol{\lambda})}\right) x_t dt\right], \\ \tfrac{d}{d\varepsilon}\mathcal{L}(C, Z+\varepsilon y,\eta) &= \mathbb{E}\left[\left(U'(Z+\varepsilon y) - \eta\pi_t^{(\boldsymbol{\lambda})}\right) y\right], \\ \tfrac{d}{d\eta}\mathcal{L}(C,Z,\eta) &= \mathbb{E}\left[\int_0^T \pi_t^{(\boldsymbol{\lambda})} C_t dt + \pi_t^{(\boldsymbol{\lambda})} Z\right] - w,\end{aligned} \tag{4.33}$$

where $x = (x_t)$ is an *arbitrary* predictable process, y is an arbitrary $\mathcal{F}_T$-measurable random variable, and η is a positive number. The first order conditions give us the solution,

$$\begin{aligned}\tfrac{d}{d\eta}|_{\eta=0}\mathcal{L}(C,Z,\eta) = 0 &\implies \eta = \eta(\boldsymbol{\lambda}), \\ \tfrac{d}{d\varepsilon}|_{\varepsilon=0}\mathcal{L}(C+\varepsilon x,Z,\eta) = 0 &\implies C_t = C_t^{(\boldsymbol{\lambda})}, \\ \tfrac{d}{d\varepsilon}|_{\varepsilon=0}\mathcal{L}(C,Z+\varepsilon y,\eta) = 0 &\implies Z = Z^{(\boldsymbol{\lambda})}.\end{aligned} \tag{4.34}$$

Note that we only checked the necessary conditions of optimality, and not the sufficient (second order) conditions.

On the other hand equation (4.21), the sufficient condition for affordability of $\left(C^{(\boldsymbol{\lambda})}, Z^{(\boldsymbol{\lambda})}\right)$, translates into equation (4.29). So, as already mentioned, if $\boldsymbol{\lambda}$ also satisfies this condition, then $\left(C^{(\boldsymbol{\lambda})}, Z^{(\boldsymbol{\lambda})}\right)$ is also the optimal consumption pair in Merton's problem. Finally equation (4.30) follows directly from proposition (4.7). □

Remark 4.10. Note that if the market is complete, i.e. $n = k$, then Σ would be invertible and the market price of risk would be $\boldsymbol{\lambda} = \Sigma^{-1}\boldsymbol{b}$. In this case equation (4.29) is automatically satisfied. This means that the "relaxed" problem in the proof of theorem (4.9) is equivalent to the Merton problem, and the solution is given by equations (4.27) and (4.30). Furthermore we may simplify equation (4.30) as follow:

$$\boldsymbol{u} = \left(\Sigma\Sigma^\top\right)^{-1}\boldsymbol{b} + \left(\Sigma^\top\right)^{-1}\boldsymbol{a}^{(C^{(\boldsymbol{\lambda})}, Z^{(\boldsymbol{\lambda})}, \boldsymbol{\lambda})}. \tag{4.35}$$

Chapter 5

The Direct Approach

In this chapter we present an new approach for tackling the problem of portfolio choice. It has the strong aspects of both existing approaches: like the stochastic control theory, it deals with the problem in its original form, and like the martingale approach, it uses probabilistic argument which can be extended to general market settings.

Recall that from equation (2.10) we have:

$$W^{(\boldsymbol{u},c)} = w\mathcal{E}\left(\int_0^{\cdot} \left\{\boldsymbol{u}_s^\top \left(\boldsymbol{\mu}_s - r_s \mathbf{1}_{n\times 1}\right) + r_s - c_s\right\} ds + \int_0^{\cdot} \boldsymbol{u}_s^\top \Sigma_s d\boldsymbol{W}_s\right). \tag{5.1}$$

The main idea of the direct approach is to use the representation above to convert Merton's problem into

$$\sup_{\boldsymbol{u}\in\boldsymbol{\mathcal{P}},\, c\in\mathcal{P}^+} \mathbb{E}\left[\int_0^T u(t, c_t W_t^{(\boldsymbol{u},c)})dt + U(W_T^{(\boldsymbol{u},c)})\right]. \tag{5.2}$$

Here $\boldsymbol{\mathcal{P}}$ is the space of n dimensional predictable vector processes, and $\mathcal{P}^+$is the space of non-negative predictable processes. Then to find a candidate for the optimal control, we may check the first order optimality conditions:

$$\begin{cases} \frac{d}{d\epsilon}|_{\epsilon=0}\mathcal{L}(\boldsymbol{u}+\epsilon\boldsymbol{v}, c) = 0, \\ \frac{d}{d\epsilon}|_{\epsilon=0}\mathcal{L}(\boldsymbol{u}, c+\epsilon y) = 0. \end{cases} \tag{5.3}$$

Where the Lagrangian $\mathcal{L}$ is of the form,

$$\mathcal{L}\left(\boldsymbol{u}, c\right) \triangleq \mathbb{E}\left[\int_0^T u(t, c_t W_t^{(\boldsymbol{u},c)})dt + U(W_T^{(\boldsymbol{u},c)})\right]. \tag{5.4}$$

Also because of the constraint $c \geq 0$, we should consider the case of zero consumption separately. In this case we define the Lagrangian as $\mathcal{L}\left(\boldsymbol{u}\right) \triangleq \mathcal{L}\left(\boldsymbol{u}, 0\right) = \mathbb{E}\left[U(W_T^{(\boldsymbol{u},c)})\right]$, and the first order condition would be $\frac{d}{d\epsilon}|_{\epsilon=0}\mathcal{L}(\boldsymbol{u}+\epsilon\boldsymbol{v}) = 0$ (we also define $W^{(\boldsymbol{u})} \triangleq W^{(\boldsymbol{u},0)}$).

In the remainder of this chapter, we will show that the first order conditions stated above can be converted into a set of equations that can be solved to obtain the optimal solutions. Namely, for the case of zero consumption we will have:

$$\boldsymbol{u}_t = \left(\Sigma_t\Sigma_t^\top\right)^{-1}\left\{\left(\boldsymbol{\mu}_t - r_t\mathbf{1}_{n\times 1}\right) + \Sigma_t \frac{\left[\boldsymbol{W}, V^{(\boldsymbol{u})}\right]_t'}{V_t^{(\boldsymbol{u})}}\right\}, \tag{5.5}$$

and for the general case we have:

$$\begin{cases} V_t^{(\boldsymbol{u},c)} = \frac{\partial}{\partial x} u\left(t, c_t W_t^{(\boldsymbol{u},c)}\right) W_t^{(\boldsymbol{u},c)}, \\ \boldsymbol{u}_t = \left(\Sigma_t \Sigma_t^\top\right)^{-1} \left\{ \left(\boldsymbol{\mu}_t - r_t \mathbf{1}_{n\times 1}\right) + \Sigma_t \frac{\left[\boldsymbol{W}, V^{(\boldsymbol{u},c)}\right]_t'}{V_t^{(\boldsymbol{u},c)}} \right\}. \end{cases} \tag{5.6}$$

Here the processes $V^{(\boldsymbol{u})}$ and $V^{(\boldsymbol{u},c)}$are defined as follows,

$$\begin{aligned} V_t^{(\boldsymbol{u})} &\triangleq \mathbb{E}_t \left[U'\left(W_T^{(\boldsymbol{u})}\right) W_T^{(\boldsymbol{u})} \right], \\ V_t^{(\boldsymbol{u},c)} &\triangleq \mathbb{E}_t \left[\int_t^T \frac{\partial}{\partial x} u\left(s, c_s W_s^{(\boldsymbol{u},c)}\right) c_s W_s^{(\boldsymbol{u},c)} ds + U'\left(W_T^{(\boldsymbol{u},c)}\right) W_T^{(\boldsymbol{u},c)} \right]. \end{aligned} \tag{5.7}$$

For the main result see theorem 5.6, and equations 5.52 and 5.55. Note that the main hurdle in implementing this approach is to find explicit expressions for $\frac{[\boldsymbol{\mathcal{W}}, V^{\boldsymbol{u}}]_t'}{V_t^{\boldsymbol{u}}}$ in the case of zero consumption, and $V^{(\boldsymbol{u},c)}$ and $\frac{[\boldsymbol{\mathcal{W}}, V^{(\boldsymbol{u},c)}]_t'}{V_t^{(\boldsymbol{u},c)}}$ for the general case.

We should mention that we have not looked at conditions which guarantee the optimality of the candidates, i.e. the counterparts of verification results in stochastic control approach (see [37, 85]) or the necessary and sufficient condition of martingale approach (see [55, 56]). As pointed out by Korn and Kraft [53], these results are crucially important and uncritical application of any method for continuous-time portfolio optimization can be misleading in the case of a stochastic opportunity set. We have proposed finding such conditions for validity of the direct approach as a future line of research (see chapter 7).

Besides these technically challenging issues, there are a number of interesting questions on the economic properties of the optimal portfolios. From (5.5) and (5.6) one sees that the optimal portfolio consists of two terms, namely,

$$\begin{aligned} \boldsymbol{u}_t^m &= \left(\Sigma_t \Sigma_t^\top\right)^{-1} \left(\boldsymbol{\mu}_t - r_t \mathbf{1}_{n\times 1}\right), \\ \boldsymbol{u}_t^h &= \left(\Sigma_t \Sigma_t^\top\right)^{-1} \Sigma_t \frac{[\boldsymbol{W}, V^*]_t'}{V_t^*}. \end{aligned} \tag{5.8}$$

The first component is known as the myopic investment strategy. It corresponds functionally to the investment policy followed by an investor in markets in which the investment opportunity set remains constant through time. The myopic portfolio is always positive for a nonzero market price of risk.

The second term is called the excess hedging demand. It represents the additional investment caused by the presence of the stochastic factor. It does not have a constant sign, for the signs of the correlation coefficient $\left(\Sigma_t \Sigma_t^\top\right)^{-1} \Sigma_t$, and the volatility of the optimal V^* process are not definite. The excess hedging demand term vanishes in the uncorrelated case; and when the volatility of the stochastic factor process is zero (the latter case can be reduced to the classical Merton one). Finally, the excess hedging demand term also vanishes for the case of logarithmic utility (see section 6.1 and [10]).

It is worth to mention that despite the nomenclature *hedging demand*, a rigorous study for the precise characterization and quantification of the risk that is not hedged has not been carried out. Indeed, in contrast to derivative valuation where the notion of imperfect hedge is well defined, such a notion has not been established in the area of investments (see [80] for a special case).

As a final observation, note that total allocation in the risky assets might become zero even if the risk premium is not zero. This phenomenon, related to the so called market participation puzzle, appears at first counter intuitive, for classical economic ideas suggest that a risk averse investor should always retain nonzero holdings in an asset that offers positive risk premium. We refer the reader to, among others, [5, 18, 44].

In the next section we will present in detail the argument used to obtain the main results of the direct approach.

5.1 Proof of the Direct approach

We start by identifying $\frac{d}{d\epsilon}|_{\epsilon=0}W_t^{(\boldsymbol{u}+\epsilon\boldsymbol{v},c)}$ and $\frac{d}{d\epsilon}|_{\epsilon=0}W_t^{(\boldsymbol{u},c+\epsilon y)}$, which in some sense, are the building blocks of our proof.

Lemma 5.1. *Let $\boldsymbol{u}$ be a self-financing strategy for the proportional consumption rate c. Then for all locally bounded predictable processes $\boldsymbol{v}$ and y we have*

$$\frac{d}{d\epsilon}|_{\epsilon=0}W_t^{(\boldsymbol{u}+\epsilon\boldsymbol{v},c)} = W_t^{(\boldsymbol{u},c)}\left(\int_0^t \boldsymbol{v}_s^T\left(\boldsymbol{\mu}_s - r_s\mathbf{1}_{n\times 1} - \Sigma_s\Sigma_s^T\boldsymbol{u}_s\right)ds + \int_0^t \boldsymbol{v}_s^T\Sigma_s d\boldsymbol{W}_s\right), \quad (5.9)$$

$$\frac{d}{d\epsilon}|_{\epsilon=0}W_t^{(\boldsymbol{u},c+\epsilon y)} = -W_t^{(\boldsymbol{u},c)}\int_0^t y_s ds. \quad (5.10)$$

Proof. Defining Y_t^ϵ as

$$Y_t^\epsilon \triangleq \int_0^t \left\{(\boldsymbol{u}_u+\epsilon\boldsymbol{v}_u)^\top(\boldsymbol{\mu}_u - r_u\mathbf{1}_{n\times 1}) + r_u - c_u\right\}du + \int_0^t(\boldsymbol{u}_u+\epsilon\boldsymbol{v}_u)^T\Sigma_u d\boldsymbol{W}_u. \quad (5.11)$$

From equation(2.10) we have $W_t^{(\boldsymbol{u}+\epsilon\boldsymbol{v},c)} = w\mathcal{E}(Y^\epsilon)_t = we^{Y_t^\epsilon - \frac{1}{2}[Y^\epsilon]_t}$. So we may conclude:

$$\frac{d}{d\epsilon}|_{\epsilon=0}W_t^{(\boldsymbol{u}+\epsilon\boldsymbol{v},c)} = W_t^{(\boldsymbol{u},c)}\frac{d}{d\epsilon}|_{\epsilon=0}\left(Y_t^\epsilon - \frac{1}{2}[Y^\epsilon]_t\right). \quad (5.12)$$

From equation(5.11), after interchanging the order of differentiation and integration, we would obtain

$$\frac{d}{d\epsilon}|_{\epsilon=0}Y_t^\epsilon = \int_0^t \boldsymbol{v}_u^T(\boldsymbol{\mu}_u - r_u\mathbf{1}_{n\times 1})du + \int_0^t \boldsymbol{v}_u^T\Sigma_u d\boldsymbol{W}_u. \quad (5.13)$$

Also from equation(5.11) it follows that

$$\begin{aligned}
[Y^\epsilon]_t &= \left[\int_0^t (\boldsymbol{u}_u+\epsilon\boldsymbol{v}_u)^T\Sigma_u d\boldsymbol{W}_u\right] \\
&= \int_0^t (\boldsymbol{u}_u+\epsilon\boldsymbol{v}_u)^T\Sigma_u d[\boldsymbol{W}_u]\Sigma_u^T(\boldsymbol{u}_u+\epsilon\boldsymbol{v}_u) \\
&= \int_0^t (\boldsymbol{u}_u+\epsilon\boldsymbol{v}_u)^T\Sigma_u(I_{k\times k}du)\Sigma_u^T(\boldsymbol{u}_u+\epsilon\boldsymbol{v}_u) \\
&= \int_0^t (\boldsymbol{u}_u+\epsilon\boldsymbol{v}_u)^T\Sigma_u\Sigma_u^T(\boldsymbol{u}_u+\epsilon\boldsymbol{v}_u)du. \quad (5.14)
\end{aligned}$$

By differentiating from the last result we would have

$$\frac{d}{d\epsilon}|_{\epsilon=0}\left[Y^{\epsilon}\right]_t = \int_0^t 2\boldsymbol{v}_u^T \Sigma_u \Sigma_u^T \boldsymbol{u}_u du. \tag{5.15}$$

Equation(5.9) follows by substituting from equation (5.13) and (5.15) into equation (5.12).

For obtaining equation (5.10) we may start by defining

$$\widetilde{Y}_t^{\epsilon} \triangleq \int_0^t \left\{\boldsymbol{u}_u^T\left(\boldsymbol{\mu}_u - r_u \mathbf{1}_{n\times 1}\right) + r_u - (c_u + \epsilon y_u)\right\} du + \int_0^t \boldsymbol{u}_u^T \Sigma_u d\boldsymbol{W}_u. \tag{5.16}$$

Similar to equation (5.12) we have:

$$\frac{d}{d\epsilon}|_{\epsilon=0} W_t^{(\boldsymbol{u},c+\epsilon y)} = W_t^{(\boldsymbol{u},c)} \frac{d}{d\epsilon}|_{\epsilon=0}\left(\widetilde{Y}_t^{\epsilon} - \frac{1}{2}\left[\widetilde{Y}^{\epsilon}\right]_t\right). \tag{5.17}$$

But $\frac{d}{d\epsilon}|_{\epsilon=0}\widetilde{Y}_t^{\epsilon} = -\int_0^t y_u du$, and $\frac{d}{d\epsilon}|_{\epsilon=0}\left[\widetilde{Y}^{\epsilon}\right]_t = 0$. Hence we obtain equation (5.10). □

In the following two lemmas, for the case of zero consumption, we find an expression for $\frac{d}{d\epsilon}|_{\epsilon=0}\mathcal{L}(\boldsymbol{u}+\epsilon\boldsymbol{v})$ in terms of the martingale $V^{(\boldsymbol{u})}$ of equation (5.51).

Lemma 5.2. *For the special case of zero consumption, define the Lagrangian as:*

$$\mathcal{L}(\boldsymbol{u}) \triangleq \mathbb{E}\left[U(W_T^{(\boldsymbol{u})})\right], \tag{5.18}$$

where $W^{(\boldsymbol{u})} = W^{(\boldsymbol{u},0)}$. Then for all locally bounded predictable processes $\boldsymbol{v}$, we have:

$$\begin{aligned}\frac{d}{d\epsilon}|_{\epsilon=0}\mathcal{L}(\boldsymbol{u}+\epsilon\boldsymbol{v}) &= \mathbb{E}\Big[U'\left(W_T^{(\boldsymbol{u})}\right) W_T^{(\boldsymbol{u})} \\ &\times \left(\int_0^T \boldsymbol{v}_s^T\left(\boldsymbol{\mu}_s - r_s\mathbf{1}_{n\times 1} - \Sigma_s\Sigma_s^T\boldsymbol{u}_s\right) ds + \int_0^T \boldsymbol{v}_s^T \Sigma_s d\boldsymbol{W}_s\right)\Big]\end{aligned} \tag{5.19}$$

Proof. By differentiating equation (5.18), and interchanging the order of differentiation with expectation, we would have

$$\begin{aligned}\frac{d}{d\epsilon}|_{\epsilon=0}\mathcal{L}(\boldsymbol{u}+\epsilon\boldsymbol{v}) &= \mathbb{E}\left[\frac{d}{d\epsilon}|_{\epsilon=0} U(W_T^{(\boldsymbol{u}+\epsilon\boldsymbol{v})})\right] \\ &= \mathbb{E}\left[U'(W_T^{(\boldsymbol{u})})\frac{d}{d\epsilon}|_{\epsilon=0} W_T^{(\boldsymbol{u}+\epsilon\boldsymbol{v})}\right].\end{aligned} \tag{5.20}$$

Now by substituting for $\frac{d}{d\epsilon}|_{\epsilon=0}W_T^{(\boldsymbol{u}+\epsilon\boldsymbol{v})}$ from equation (5.9) we will get the result. □

Lemma 5.3. *For the special case of zero consumption, we define the martingale$V^{(\boldsymbol{u})}$ as:*

$$V_t^{(\boldsymbol{u})} \triangleq \mathbb{E}_t\left[U'\left(W_T^{(\boldsymbol{u})}\right) W_T^{(\boldsymbol{u})}\right]. \tag{5.21}$$

Then for all locally bounded predictable processes $\boldsymbol{v}$, we have

$$\frac{d}{d\epsilon}|_{\epsilon=0}\mathcal{L}(\boldsymbol{u}+\epsilon\boldsymbol{v}) = \mathbb{E}\left[\int_0^T V_s^{(\boldsymbol{u})}\boldsymbol{v}_s^T\left(\boldsymbol{\mu}_s - r_s\mathbf{1}_{2\times 1} - \Sigma_s\Sigma_s^T\boldsymbol{u}_s\right) ds + \int_0^T \boldsymbol{v}_s^T\Sigma_s d\left[\boldsymbol{W}, V^{(\boldsymbol{u})}\right]_s\right]. \tag{5.22}$$

Proof. By considering the fact that $V_T^{\boldsymbol{u}} = U'\left(W_T^{(\boldsymbol{u})}\right) W_T^{(\boldsymbol{u})}$, equation (5.19) can be rewritten as

$$\frac{d}{d\epsilon}|_{\epsilon=0}\mathcal{L}(\boldsymbol{u}+\epsilon\boldsymbol{v}) = \mathbb{E}\left[V_T^{\boldsymbol{u}}\left(\int_0^T \boldsymbol{v}_s^T\left(\boldsymbol{\mu}_s - r_s\mathbf{1}_{2\times 1} - \Sigma_s\Sigma_s^T\boldsymbol{u}_s\right)ds + \int_0^T \boldsymbol{v}_s^T\Sigma_s d\boldsymbol{W}_s\right)\right]. \quad (5.23)$$

By integration by parts we have:

$$\begin{aligned} & V_T^{\boldsymbol{u}}\left(\int_0^T \boldsymbol{v}_s^T\left(\boldsymbol{\mu}_s - r_s\mathbf{1}_{2\times 1} - \Sigma_s\Sigma_s^T\boldsymbol{u}_s\right)ds + \int_0^T \boldsymbol{v}_s^T\Sigma_s d\boldsymbol{W}_s\right) \\ = & \int_0^T V_s^{\boldsymbol{u}}\boldsymbol{v}_s^T\left(\boldsymbol{\mu}_s - r_s\mathbf{1}_{2\times 1} - \Sigma_s\Sigma_s^T\boldsymbol{u}_s\right)ds + \int_0^T V_s^{\boldsymbol{u}}\boldsymbol{v}_s^T\Sigma_s d\boldsymbol{W}_s \\ & + \int_0^T \left(\int_0^t \boldsymbol{v}_s^T\left(\boldsymbol{\mu}_s - r_s\mathbf{1}_{2\times 1} - \Sigma_s\Sigma_s^T\boldsymbol{u}_s\right)ds + \int_0^t \boldsymbol{v}_s^T\Sigma_s d\boldsymbol{W}_s\right) dV_t^{\boldsymbol{u}} \\ & + \int_0^T \boldsymbol{v}_s^T\Sigma_s d\left[\boldsymbol{W}, V^{\boldsymbol{u}}\right]_s. \end{aligned} \quad (5.24)$$

Note that, subject to some technical conditions, the second and third terms in the right-hand-side are zero-mean martingales. Now by taking expectation we will get the result. □

The following two lemmas are the counterparts of lemmas 5.2 and 5.3, for the case of non-zero consumption.

Lemma 5.4. *For the general case with consumption, define the Lagrangian as:*

$$\mathcal{L}(\boldsymbol{u}, c) \triangleq \mathbb{E}\left[\int_0^T u(t, c_t W_t^{(\boldsymbol{u},c)})dt + U(W_T^{(\boldsymbol{u},c)})\right]. \quad (5.25)$$

Then for all locally bounded predictable processes $\boldsymbol{v}$, *we would have:*

$$\begin{aligned} & \frac{d}{d\epsilon}|_{\epsilon=0}\mathcal{L}(\boldsymbol{u}+\epsilon\boldsymbol{v}, c) \\ = & \mathbb{E}\left[\int_0^T \frac{\partial}{\partial x}u\left(t, c_t W_t^{(\boldsymbol{u},c)}\right) c_t W_t^{(\boldsymbol{u},c)}\left(\int_0^t \boldsymbol{v}_s^T\left(\boldsymbol{\mu}_s - r_s\mathbf{1}_{2\times 1} - \Sigma_s\Sigma_s^T\boldsymbol{u}_s\right)ds + \int_0^t \boldsymbol{v}_s^T\Sigma_s d\boldsymbol{W}_s\right)dt\right] \\ & + \mathbb{E}\left[U'\left(W_T^{(\boldsymbol{u},c)}\right) W_T^{(\boldsymbol{u},c)}\left(\int_0^T \boldsymbol{v}_s^T\left(\boldsymbol{\mu}_s - r_s\mathbf{1}_{2\times 1} - \Sigma_s\Sigma_s^T\boldsymbol{u}_s\right)ds + \int_0^T \boldsymbol{v}_s^T\Sigma_s d\boldsymbol{W}_s\right)\right], \end{aligned} \quad (5.26)$$

and for all locally bounded predictable processes y, *we have:*

$$\begin{aligned} & \frac{d}{d\epsilon}|_{\epsilon=0}\mathcal{L}(\boldsymbol{u}, c+\epsilon y) \\ = & \mathbb{E}\left[\int_0^T \frac{\partial}{\partial x}u\left(t, c_t W_t^{(\boldsymbol{u},c)}\right) W_t^{(\boldsymbol{u},c)}\left(y_t - c_t\int_0^t y_s ds\right)dt\right] \\ & - \mathbb{E}\left[U'\left(W_T^{(\boldsymbol{u},c)}\right) W_T^{(\boldsymbol{u},c)}\int_0^T y_s ds\right]. \end{aligned} \quad (5.27)$$

Proof. By differentiating equation (5.25), and interchanging the order of differentiation with expectation and then with integration, we would have

$$\begin{aligned}\frac{d}{d\epsilon}|_{\epsilon=0}\mathcal{L}(\boldsymbol{u}+\epsilon\boldsymbol{v},c) &= \mathbb{E}\left[\int_0^T \frac{d}{d\epsilon}|_{\epsilon=0}u(t,c_tW_t^{(\boldsymbol{u}+\epsilon\boldsymbol{v},c)})dt + \frac{d}{d\epsilon}|_{\epsilon=0}U(W_T^{(\boldsymbol{u}+\epsilon\boldsymbol{v},c)})\right] \\ &= \mathbb{E}\left[\int_0^T \frac{\partial}{\partial x}u\left(t,c_tW_t^{(\boldsymbol{u},c)}\right)c_t\left(\frac{d}{d\epsilon}|_{\epsilon=0}W_t^{(\boldsymbol{u}+\epsilon\boldsymbol{v},c)}\right)dt\right] \\ &\quad +\mathbb{E}\left[U'(W_T^{(\boldsymbol{u},c)})\frac{d}{d\epsilon}|_{\epsilon=0}W_T^{(\boldsymbol{u}+\epsilon\boldsymbol{v},c)}\right]. \end{aligned} \tag{5.28}$$

Now by substituting for $\frac{d}{d\epsilon}|_{\epsilon=0}W_t^{(\boldsymbol{u}+\epsilon\boldsymbol{v},c)}$ and $\frac{d}{d\epsilon}|_{\epsilon=0}W_T^{(\boldsymbol{u}+\epsilon\boldsymbol{v},c)}$from equation (5.9) we will get equation (5.26).

For obtaining equation (5.27), by differentiating equation (5.25) we would have:

$$\begin{aligned}\frac{d}{d\epsilon}|_{\epsilon=0}\mathcal{L}(\boldsymbol{u},c+\epsilon y) &= \mathbb{E}\left[\int_0^T \frac{d}{d\epsilon}|_{\epsilon=0}u(t,(c_t+\epsilon y_t)\,W_t^{(\boldsymbol{u},c+\epsilon y)})dt + \frac{d}{d\epsilon}|_{\epsilon=0}U(W_T^{(\boldsymbol{u},c+\epsilon y)})\right] \\ &= \mathbb{E}\left[\int_0^T \frac{\partial}{\partial x}u\left(t,c_tW_t^{(\boldsymbol{u},c)}\right)c_t\left(\frac{d}{d\epsilon}|_{\epsilon=0}W_t^{(\boldsymbol{u},c+\epsilon y)}\right)dt\right] \\ &\quad +\mathbb{E}\left[\int_0^T \frac{\partial}{\partial x}u\left(t,c_tW_t^{(\boldsymbol{u},c)}\right)y_tW_t^{(\boldsymbol{u},c+\epsilon y)}dt\right] \qquad (5.29) \\ &\quad +\mathbb{E}\left[U'(W_T^{(\boldsymbol{u},c)})\frac{d}{d\epsilon}|_{\epsilon=0}W_T^{(\boldsymbol{u},c+\epsilon y)}\right]. \qquad (5.30)\end{aligned}$$

Now substituting for $\frac{d}{d\epsilon}|_{\epsilon=0}W_t^{(\boldsymbol{u},c+\epsilon y)}$ and $\frac{d}{d\epsilon}|_{\epsilon=0}W_T^{(\boldsymbol{u},c+\epsilon y)}$ from equation(5.10) would result in equation (5.27). □

Lemma 5.5. *Define the process*$V^{(\boldsymbol{u},c)}$ *by*

$$V_s^{(\boldsymbol{u},c)} \triangleq \mathbb{E}_s\left[\int_s^T \frac{\partial}{\partial x}u\left(t,c_tW_t^{(\boldsymbol{u},c)}\right)c_tW_t^{(\boldsymbol{u},c)}dt + U'\left(W_T^{(\boldsymbol{u})}\right)W_T^{(\boldsymbol{u})}\right]. \tag{5.31}$$

Then for all locally bounded predictable processes $\boldsymbol{v}$*, we would have:*

$$\frac{d}{d\epsilon}|_{\epsilon=0}\mathcal{L}(\boldsymbol{u}+\epsilon\boldsymbol{v},c) = \mathbb{E}\left[\int_0^T V_s^{(\boldsymbol{u},c)}\boldsymbol{v}_s^T\left(\boldsymbol{\mu}_s - r_s\mathbf{1}_{2\times 1} - \Sigma_s\Sigma_s^T\boldsymbol{u}_s\right)ds + \int_0^T \boldsymbol{v}_s^T\Sigma_s d\left[\boldsymbol{W},V^{(\boldsymbol{u},c)}\right]_s\right], \tag{5.32}$$

and for all locally bounded predictable processes y*, we have:*

$$\frac{d}{d\epsilon}|_{\epsilon=0}\mathcal{L}(\boldsymbol{u},c+\epsilon y) = \mathbb{E}\left[\int_0^T\left(\frac{\partial}{\partial x}u\left(t,c_tW_t^{(\boldsymbol{u},c)}\right)W_t^{(\boldsymbol{u},c)} - V_t^{(\boldsymbol{u},c)}\right)y_tdt\right]. \tag{5.33}$$

Proof. To derive equation (5.33), we note that by equation (5.27)

$$\frac{d}{d\epsilon}|_{\epsilon=0}\mathcal{L}(\boldsymbol{u},c+\epsilon y) = \mathbb{E}\left[\int_0^T \frac{\partial}{\partial x}u\left(t,c_tW_t^{(\boldsymbol{u},c)}\right)W_t^{(\boldsymbol{u},c)}y_tdt\right] - I - II, \tag{5.34}$$

where

$$\begin{aligned} I &\triangleq \mathbb{E}\left[\int_0^T \frac{\partial}{\partial x}u\left(t, c_t W_t^{(\boldsymbol{u},c)}\right) c_t W_t^{(\boldsymbol{u},c)} \left(\int_0^t y_s ds\right) dt\right] \\ II &\triangleq \mathbb{E}\left[U'\left(W_T^{(\boldsymbol{u},c)}\right) W_T^{(\boldsymbol{u},c)} \left(\int_0^T y_s ds\right)\right]. \end{aligned} \tag{5.35}$$

For I we may write

$$\begin{aligned} I &= \int_0^T \int_0^t \mathbb{E}\left[\frac{\partial}{\partial x}u\left(t, c_t W_t^{(\boldsymbol{u},c)}\right) c_t W_t^{(\boldsymbol{u},c)} y_s\right] dsdt \\ &= \int_0^T \int_s^T \mathbb{E}\left[\frac{\partial}{\partial x}u\left(t, c_t W_t^{(\boldsymbol{u},c)}\right) c_t W_t^{(\boldsymbol{u},c)} y_s\right] dtds \\ &= \int_0^T \int_s^T \mathbb{E}\left[\mathbb{E}_s\left[\frac{\partial}{\partial x}u\left(t, c_t W_t^{(\boldsymbol{u},c)}\right) c_t W_t^{(\boldsymbol{u},c)} y_s\right]\right] dtds \\ &= \mathbb{E}\left[\int_0^T y_s \mathbb{E}_s\left[\int_s^T \frac{\partial}{\partial x}u\left(t, c_t W_t^{(\boldsymbol{u},c)}\right) c_t W_t^{(\boldsymbol{u},c)} dt\right] ds\right]. \end{aligned} \tag{5.36}$$

For handling II, first we define a martingale $G_s^{(\boldsymbol{u},c)} \triangleq \mathbb{E}_s\left[U'\left(W_T^{(\boldsymbol{u},c)}\right) W_T^{(\boldsymbol{u},c)}\right]$. Then applying integration by parts to the definition of II gives us

$$\begin{aligned} II &= \mathbb{E}\left[G_T^{(\boldsymbol{u},c)} \left(\int_0^T y_s ds\right)\right] \\ &= \mathbb{E}\left[\int_0^T y_s G_s^{(\boldsymbol{u},c)} ds + \int_0^T \left(\int_0^t y_s ds\right) dG_t^{(\boldsymbol{u},c)}\right] \\ &= \mathbb{E}\left[\int_0^T y_s G_s^{(\boldsymbol{u},c)} ds\right], \end{aligned} \tag{5.37}$$

where we used the fact that, subject to some technical conditions, the second term in the second line is a martingale so its expectation vanishes. Now substituting for I and II in equation (5.34) will result in equation (5.33).

Similarly, to derive equation (5.32), we will start by equation (5.26)

$$\frac{d}{d\epsilon}|_{\epsilon=0} \boldsymbol{\mathcal{L}}\left(\boldsymbol{u} + \epsilon \boldsymbol{v}, c\right) = I + II + III, \tag{5.38}$$

where

$$I \triangleq \mathbb{E}\left[\int_0^T \frac{\partial}{\partial x}u\left(t, c_t W_t^{(\boldsymbol{u},c)}\right) c_t W_t^{(\boldsymbol{u},c)} \left(\int_0^t \boldsymbol{v}_s^T \left(\boldsymbol{\mu}_s - r_s \mathbf{1}_{2\times 1} - \Sigma_s \Sigma_s^T \boldsymbol{u}_s\right) ds\right) dt\right] \tag{5.39}$$

$$II \triangleq \mathbb{E}\left[\int_0^T \frac{\partial}{\partial x}u\left(t, c_t W_t^{(\boldsymbol{u},c)}\right) c_t W_t^{(\boldsymbol{u},c)} \left(\int_0^t \boldsymbol{v}_s^T \Sigma_s d\boldsymbol{W}_s\right) dt\right] \tag{5.40}$$

$$\begin{aligned} III \triangleq\ & \mathbb{E}\Bigg[U'\left(W_T^{(\boldsymbol{u},c)}\right) W_T^{(\boldsymbol{u},c)} \\ & \times \left(\int_0^T \boldsymbol{v}_s^T \left(\boldsymbol{\mu}_s - r_s \mathbf{1}_{2\times 1} - \Sigma_s \Sigma_s^T \boldsymbol{u}_s\right) ds + \int_0^T \boldsymbol{v}_s^T \Sigma_s d\boldsymbol{W}_s\right)\Bigg]. \end{aligned} \tag{5.41}$$

Expression III can be handled by following the same argument as in the proof of lemma 5.3, which gives us

$$III = \mathbb{E}\left[\int_0^T \widetilde{V}_s^{(\boldsymbol{u},c)} \boldsymbol{v}_s^T \left(\boldsymbol{\mu}_s - r_s \mathbf{1}_{2\times 1} - \Sigma_s \Sigma_s^T \boldsymbol{u}_s\right) ds + \int_0^T \boldsymbol{v}_s^T \Sigma_s d\left[\boldsymbol{W}, \widetilde{V}^{(\boldsymbol{u},c)}\right]_s\right], \tag{5.42}$$

where

$$\widetilde{V}_s^{(\boldsymbol{u},c)} \triangleq \mathbb{E}_s\left[U'\left(W_T^{(\boldsymbol{u},c)}\right) W_T^{(\boldsymbol{u},c)}\right]. \tag{5.43}$$

For handling I, we follow a calculation similar to equation (5.36)

$$\begin{aligned}
I &= \int_0^T \int_0^t \mathbb{E}\left[\frac{\partial}{\partial x} u\left(t, c_t W_t^{(\boldsymbol{u},c)}\right) c_t W_t^{(\boldsymbol{u},c)} \boldsymbol{v}_s^T \left(\boldsymbol{\mu}_s - r_s \mathbf{1}_{2\times 1} - \Sigma_s \Sigma_s^T \boldsymbol{u}_s\right)\right] ds dt \\
&= \int_0^T \int_s^T \mathbb{E}\left[\frac{\partial}{\partial x} u\left(t, c_t W_t^{(\boldsymbol{u},c)}\right) c_t W_t^{(\boldsymbol{u},c)} \boldsymbol{v}_s^T \left(\boldsymbol{\mu}_s - r_s \mathbf{1}_{2\times 1} - \Sigma_s \Sigma_s^T \boldsymbol{u}_s\right)\right] dt ds \\
&= \int_0^T \int_s^T \mathbb{E}\left[\mathbb{E}_s\left[\frac{\partial}{\partial x} u\left(t, c_t W_t^{(\boldsymbol{u},c)}\right) c_t W_t^{(\boldsymbol{u},c)} \boldsymbol{v}_s^T \left(\boldsymbol{\mu}_s - r_s \mathbf{1}_{2\times 1} - \Sigma_s \Sigma_s^T \boldsymbol{u}_s\right)\right]\right] dt ds \\
&= \mathbb{E}\left[\int_0^T \mathbb{E}_s\left[\int_s^T \frac{\partial}{\partial x} u\left(t, c_t W_t^{(\boldsymbol{u},c)}\right) c_t W_t^{(\boldsymbol{u},c)} dt\right] \boldsymbol{v}_s^T \left(\boldsymbol{\mu}_s - r_s \mathbf{1}_{2\times 1} - \Sigma_s \Sigma_s^T \boldsymbol{u}_s\right) ds\right]
\end{aligned} \tag{5.44}$$

For handling II, first we define:

$$V_s^{(\boldsymbol{u},c,t)} \triangleq \mathbb{E}_s\left[\frac{\partial}{\partial x} u\left(t, c_t W_t^{(\boldsymbol{u},c)}\right) c_t W_t^{(\boldsymbol{u},c)}\right] \tag{5.45}$$

Now integration by parts would give us

$$\begin{aligned}
&\mathbb{E}\left[\frac{\partial}{\partial x} u\left(t, c_t W_t^{(\boldsymbol{u},c)}\right) c_t W_t^{(\boldsymbol{u},c)} \left(\int_0^t \boldsymbol{v}_s^T \Sigma_s d\boldsymbol{W}_s\right)\right] \\
&= \mathbb{E}\left[V_t^{(\boldsymbol{u},c,t)} \left(\int_0^t \boldsymbol{v}_s^T \Sigma_s d\boldsymbol{W}_s\right)\right] \\
&= \mathbb{E}\left[\int_0^t V_s^{(\boldsymbol{u},c,t)} \boldsymbol{v}_s^T \Sigma_s d\boldsymbol{W}_s + \int_0^t \left(\int_0^s \boldsymbol{v}_u^T \Sigma_u d\boldsymbol{W}_u\right) dV_s^{(\boldsymbol{u},c,t)} + \int_0^t \boldsymbol{v}_s^T \Sigma_s d\left[\boldsymbol{W}, V^{(\boldsymbol{u},c,t)}\right]_s\right] \\
&= \mathbb{E}\left[\int_0^t \boldsymbol{v}_s^T \Sigma_s d\left[\boldsymbol{W}, V^{(\boldsymbol{u},c,t)}\right]_s\right],
\end{aligned} \tag{5.46}$$

where we assume that the first two integrals in the third line are zero mean martingales. We may now proceed as follow:

$$\begin{aligned} II &= \int_0^T \mathbb{E}\left[\frac{\partial}{\partial x}u\left(t, c_t W_t^{(\boldsymbol{u},c)}\right) c_t W_t^{(\boldsymbol{u},c)} \left(\int_0^t \boldsymbol{v}_s^T \Sigma_s d\boldsymbol{W}_s\right)\right] dt \\ &= \int_0^T \mathbb{E}\left[\int_0^t \boldsymbol{v}_s^T \Sigma_s d\left[\boldsymbol{W}, V_\cdot^{(\boldsymbol{u},c,t)}\right]_s\right] dt \\ &= \mathbb{E}\left[\int_0^T \int_0^t \boldsymbol{v}_s^T \Sigma_s \left[\boldsymbol{W}, V_\cdot^{(\boldsymbol{u},c,t)}\right]_s' dsdt\right] \\ &= \mathbb{E}\left[\int_0^T \int_s^T \boldsymbol{v}_s^T \Sigma_s \left[\boldsymbol{W}, V_\cdot^{(\boldsymbol{u},c,t)}\right]_s' dtds\right] \\ &= \mathbb{E}\left[\int_0^T \boldsymbol{v}_s^T \Sigma_s \left[\boldsymbol{W}, \int_\cdot^T V_\cdot^{(\boldsymbol{u},c,t)} dt\right]_s' ds\right] \\ &= \mathbb{E}\left[\int_0^T \boldsymbol{v}_s^T \Sigma_s d\left[\boldsymbol{W}, \int_\cdot^T V_\cdot^{(\boldsymbol{u},c,t)} dt\right]_s\right]. \end{aligned} \tag{5.47}$$

Finally we put everything back together to obtain equation (5.32):

$$\begin{aligned} \frac{d}{d\epsilon}|_{\epsilon=0}\mathcal{L}\left(\boldsymbol{u}+\epsilon\boldsymbol{v}, c\right) &= I + II + III \\ &= \mathbb{E}\left[\int_0^T \mathbb{E}_s\left[\int_s^T \frac{\partial}{\partial x}u\left(t, c_t W_t^{(\boldsymbol{u},c)}\right) c_t W_t^{(\boldsymbol{u},c)} dt\right] \right. \\ &\qquad \left. \times\, \boldsymbol{v}_s^T \left(\boldsymbol{\mu}_s - r_s \mathbf{1}_{2\times 1} - \Sigma_s \Sigma_s^T \boldsymbol{u}_s\right) ds\right] \qquad (5.48) \\ &\quad + \mathbb{E}\left[\int_0^T \boldsymbol{v}_s^T \Sigma_s d\left[\boldsymbol{W}, \int_\cdot^T V_\cdot^{(\boldsymbol{u},c,t)} dt\right]_s\right] \\ &\quad + \mathbb{E}\left[\int_0^T \widetilde{V}_s^{(\boldsymbol{u},c)} \boldsymbol{v}_s^\top \left(\boldsymbol{\mu}_s - r_s \mathbf{1}_{2\times 1} - \Sigma_s \Sigma_s^T \boldsymbol{u}_s\right) ds\right] \\ &\quad + \mathbb{E}\left[\int_0^T \boldsymbol{v}_s^\top \Sigma_s d\left[\boldsymbol{W}, \widetilde{V}^{(\boldsymbol{u},c)}\right]_s\right] \qquad (5.49) \\ &= \mathbb{E}\left[\int_0^T \mathbb{E}_s\left[\int_s^T \frac{\partial}{\partial x}u\left(t, c_t W_t^{(\boldsymbol{u},c)}\right) c_t W_t^{(\boldsymbol{u},c)} dt + U'\left(W_T^{(\boldsymbol{u},c)}\right) W_T^{(\boldsymbol{u},c)}\right] \right. \\ &\qquad \left. \times \boldsymbol{v}_s^T \left(\boldsymbol{\mu}_s - r_s \mathbf{1}_{2\times 1} - \Sigma_s \Sigma_s^T \boldsymbol{u}_s\right) ds\right] \\ &\quad + \mathbb{E}\left[\int_0^T \boldsymbol{v}_s^T \Sigma_s \left(d\left[\boldsymbol{W}, \int_\cdot^T V_\cdot^{(\boldsymbol{u},c,t)} dt\right]_s + d\left[\boldsymbol{W}, \widetilde{V}^{(\boldsymbol{u},c)}\right]_s\right)\right] \\ &= \mathbb{E}\left[\int_0^T V_s^{(\boldsymbol{u},c)} \boldsymbol{v}_s^T \left(\boldsymbol{\mu}_s - r_s \mathbf{1}_{2\times 1} - \Sigma_s \Sigma_s^T \boldsymbol{u}_s\right) ds\right] \\ &\quad + \mathbb{E}\left[\int_0^T \boldsymbol{v}_s^T \Sigma_s d\left[\boldsymbol{W}, V^{(\boldsymbol{u},c)}\right]_s\right], \qquad (5.50) \end{aligned}$$

Where in the last step we used the fact that $V_s^{(\boldsymbol{u},c)} = \int_s^T V_s^{(\boldsymbol{u},c,t)} dt + \widetilde{V}_s^{(\boldsymbol{u},c)}$. □

Finally we are ready to prove the main result.

Theorem 5.6. *(the Direct Approach) Consider problem 2.9. For the case of zero consumption, define the martingale* $V^{(\boldsymbol{u})}$ *by*

$$V_t^{(\boldsymbol{u})} \triangleq \mathbb{E}_t \left[U'\left(W_T^{(\boldsymbol{u})} \right) W_T^{(\boldsymbol{u})} \right]. \tag{5.51}$$

Then a self-financing strategy $\boldsymbol{u}$ *(with no consumption) satisfies the first-order optimality conditions, i.e.* $\frac{d}{d\epsilon}|_{\epsilon=0}\mathcal{L}(\boldsymbol{u}+\epsilon\boldsymbol{v}) = 0$ *for all bounded predictable process* $\boldsymbol{v}$*, if and only if*

$$\boldsymbol{u}_t = \left(\Sigma_t \Sigma_t^T \right)^{-1} \left\{ (\boldsymbol{\mu}_t - r_t \mathbf{1}_{n\times 1}) + \Sigma_t \frac{[\boldsymbol{W}, V^{\boldsymbol{u}}]_t'}{V_t^{\boldsymbol{u}}} \right\}. \tag{5.52}$$

For the general case (i.e. nonzero consumption) define the process$V^{(\boldsymbol{u},c)}$ *by*

$$V_t^{(\boldsymbol{u},c)} \triangleq \mathbb{E}_t \left[\int_t^T \frac{\partial}{\partial x} u\left(s, c_s W_s^{(\boldsymbol{u},c)}\right) c_s W_s^{(\boldsymbol{u},c)} ds + U'\left(W_T^{(\boldsymbol{u},c)} \right) W_T^{(\boldsymbol{u},c)} \right]. \tag{5.53}$$

Then the pair $(\boldsymbol{u}, c)$ *satisfies the first-order optimality conditions, i.e.*

$$\begin{cases} \frac{d}{d\epsilon}|_{\epsilon=0}\mathcal{L}(\boldsymbol{u}+\epsilon\boldsymbol{v}, c) = 0 \\ \frac{d}{d\epsilon}|_{\epsilon=0}\mathcal{L}(\boldsymbol{u}, c+\epsilon y) = 0 \end{cases} \tag{5.54}$$

for all bounded predictable process $\boldsymbol{v}$ *and* y*, if and only if*

$$\begin{cases} V_t^{(\boldsymbol{u},c)} = \frac{\partial}{\partial x} u\left(t, c_t W_t^{(\boldsymbol{u},c)}\right) W_t^{(\boldsymbol{u},c)}, \\ \boldsymbol{u}_t = \left(\Sigma_t \Sigma_t^T \right)^{-1} \left\{ (\boldsymbol{\mu}_t - r_t \mathbf{1}_{n\times 1}) + \Sigma_t \frac{\left[\boldsymbol{W}, V^{(\boldsymbol{u},c)}\right]_t'}{V_t^{(\boldsymbol{u},c)}} \right\}. \end{cases} \tag{5.55}$$

Proof. First consider the case of zero consumption. By lemma (5.3) the first order condition becomes

$$\mathbb{E}\left[\int_0^T \boldsymbol{v}_s^T \left\{ V_s^{\boldsymbol{u}} \left(\boldsymbol{\mu}_s - r_s \mathbf{1}_{2\times 1} - \Sigma_s \Sigma_s^T \boldsymbol{u}_s \right) + \Sigma_s \left[\boldsymbol{W}, V^{\boldsymbol{u}} \right]_s' \right\} ds \right] = 0, \tag{5.56}$$

for all bounded *predictable* processes $\boldsymbol{v}$. Since the coefficient in the braces is predictable, we may conclude that it must vanish:

$$V_s^{\boldsymbol{u}} \left(\boldsymbol{\mu}_s - r_s \mathbf{1}_{2\times 1} - \Sigma_s \Sigma_s^T \boldsymbol{u}_s \right) + \Sigma_s \left[\boldsymbol{W}, V^{\boldsymbol{u}} \right]_s' = 0, \text{ for all } 0 < s < T. \tag{5.57}$$

This last result is equivalent to equation (5.52).

For the general case (with nonzero consumption) we use lemma (5.5) to rewrite the first order conditions as:

$$\mathbb{E}\left[\int_0^T \left\{ \frac{\partial}{\partial x} u\left(t, c_t W_t^{(\boldsymbol{u},c)}\right) W_t^{(\boldsymbol{u},c)} - V_t^{(\boldsymbol{u},c)} \right\} y_t dt \right] = 0, \tag{5.58}$$

$$\mathbb{E}\left[\int_0^T \boldsymbol{v}_s^T \left\{ V_s^{(\boldsymbol{u},c)} \left(\boldsymbol{\mu}_s - r_s \mathbf{1}_{2\times 1} - \Sigma_s \Sigma_s^T \boldsymbol{u}_s \right) + \Sigma_s \left[\boldsymbol{W}, V^{(\boldsymbol{u},c)} \right]_s' \right\} ds \right] = 0, \tag{5.59}$$

for all bounded *predictable* process $\boldsymbol{v}$ and y. Again since all terms in the braces are predictable, they must vanish:

$$\begin{gathered} \frac{\partial}{\partial x} u\left(t, c_t W_t^{(\boldsymbol{u},c)}\right) W_t^{(\boldsymbol{u},c)} - V_t^{(\boldsymbol{u},c)} = 0, \text{ for all } 0 < t < T, \\ V_s^{(\boldsymbol{u},c)} \left(\boldsymbol{\mu}_s - r_s \mathbf{1}_{2\times 1} - \Sigma_s \Sigma_s^T \boldsymbol{u}_s \right) + \Sigma_s \left[\boldsymbol{W}, V^{(\boldsymbol{u},c)} \right]_s' = 0, \text{ for all } 0 < s < T. \end{gathered} \tag{5.60}$$

This last system is equivalent to equation (5.55). □

Chapter 6

Examples

The most frequently used utilities are the exponential, power and logarithmic ones. Exponential case case has been extensively studied not only in optimal investment models but, also, in indifference pricing where valuation is done primarily under exponential preferences (see [19] for a concise collection of relevant references). However, a well known criticism of the exponential utility is that the optimal portfolio does not depend on the investors wealth. While this property might be desirable in asset equilibrium pricing, it appears to be problematic and counter intuitive for investment problems. Here we focus on logarithmic and power utilities.

6.1 Logarithmic Utility

In this section, we consider the general market of definition 2.1 and the logarithmic utility functional of the following form,

$$J(C,Z) \triangleq \mathbb{E}\left[\int_0^T \log\left(C_t\right)dt + \log\left(Z\right)\right]. \tag{6.1}$$

The logarithmic utility plays a special role in portfolio choice. As it will be shown in this section, the optimal portfolio is myopic, namely,

$$\boldsymbol{u} = \left(\Sigma\Sigma^T\right)^{-1}\left(\boldsymbol{\mu} - r\mathbf{1}_{n\times 1}\right), \tag{6.2}$$

This is a well-known fact, and the associated myopic portfolio, also know as the *growth optimal portfolio*, has been extensively studied in the general semimartingale market settings (see, for example, [4] and [45]). The associated optimal wealth is the so-called "numeraire portfolio". It has also been extensively studied, for it is the numeraire with regards to which all wealth processes are supermartingales under the historical measure (see, among others, [40] and [41]).

Using the general results of chapter 4 and 5 would be an overkill for the case of logarithmic utility, since there is a much more simpler way. Actually the idea of the direct approach stem from this solution method. Note that from equation(2.10) we have $W_t^{(\boldsymbol{u},c)} = w\mathcal{E}\left(Y\right)_t = we^{Y_t - \frac{1}{2}[Y]_t}$, where,

$$\begin{gathered} Y_t \triangleq \int_0^t \left\{\boldsymbol{u}_s^\top\left(\boldsymbol{\mu}_s - r_s\mathbf{1}_{n\times 1}\right) + r_s - c_s\right\}ds + \int_0^t \boldsymbol{u}_s^\top\Sigma_s d\boldsymbol{W}_s, \\ [Y]_t = \int_0^t \boldsymbol{u}_s^\top\Sigma_s\Sigma_s^\top\boldsymbol{u}_s ds. \end{gathered} \tag{6.3}$$

The optimum criterion can be expanded as follow,

$$\mathbb{E}\left[\int_0^T \log(c_t W_t^{(\boldsymbol{u},c)})dt + \log(W_T^{(\boldsymbol{u},c)})\right] \tag{6.4}$$

$$= \mathbb{E}\left[\int_0^T \log(c_t)\,dt + \int_0^T \log(W_t^{(\boldsymbol{u},c)})dt + \log\left(W_T^{(\boldsymbol{u},c)}\right)\right]. \tag{6.5}$$

Now we may with the second term above to obtain,

$$\begin{aligned}
& \mathbb{E}\left[\log(W_T^{(\boldsymbol{u},c)})\right] \\
= \ & \log(w) + \mathbb{E}\left[Y_T - \frac{1}{2}[Y]_T\right] \\
= \ & \log(w) + \mathbb{E}\left[\int_0^T \left\{-\frac{1}{2}\boldsymbol{u}_s^\top \Sigma_s \Sigma_s^\top \boldsymbol{u}_s + \boldsymbol{u}_s^\top(\boldsymbol{\mu}_s - r_s \mathbf{1}_{n\times 1}) + r_s - c_s\right\} ds\right].
\end{aligned} \tag{6.6}$$

The third term can also be written in the form,

$$\begin{aligned}
& \mathbb{E}\left[\int_0^T \log(W_t^{(\boldsymbol{u},c)})dt\right] - T\log(w) \\
= \ & \mathbb{E}\left[\int_0^T Y_t dt - \int_0^T \frac{1}{2}[Y]_t\,dt\right] \\
= \ & \mathbb{E}\left[\int_0^T \int_0^t \left\{-\frac{1}{2}\boldsymbol{u}_s^\top \Sigma_s \Sigma_s^\top \boldsymbol{u}_s + \boldsymbol{u}_s^\top(\boldsymbol{\mu}_s - r_s \mathbf{1}_{n\times 1}) + r_s - c_s\right\} dsdt\right] \\
& + \mathbb{E}\left[\int_0^T \left(\int_0^t \boldsymbol{u}_s^\top \Sigma_s d\boldsymbol{W}_s\right) dt\right] \\
= \ & \mathbb{E}\left[\int_0^T \int_s^T \left\{-\frac{1}{2}\boldsymbol{u}_s^\top \Sigma_s \Sigma_s^\top \boldsymbol{u}_s + \boldsymbol{u}_s^\top(\boldsymbol{\mu}_s - r_s \mathbf{1}_{n\times 1}) + r_s - c_s\right\} dtds\right] \\
& + \int_0^T \mathbb{E}\left[\int_0^t \boldsymbol{u}_s^\top \Sigma_s d\boldsymbol{W}_s\right] dt \\
= \ & \mathbb{E}\left[\int_0^T (T-s)\left\{-\frac{1}{2}\boldsymbol{u}_s^\top \Sigma_s \Sigma_s^\top \boldsymbol{u}_s + \boldsymbol{u}_s^\top(\boldsymbol{\mu}_s - r_s \mathbf{1}_{n\times 1}) + r_s - c_s\right\} ds\right],
\end{aligned} \tag{6.7}$$

where in the last step we assume $\int_0^t \boldsymbol{u}_s^\top \Sigma_s d\boldsymbol{W}_s$ to be a martingale. Substituting back the last two results in the optimal criterion gives us,

$$\begin{aligned}
& \mathbb{E}\left[\int_0^T \log(c_t W_t^{(\boldsymbol{u},c)})dt + \log(W_T^{(\boldsymbol{u},c)})\right] \\
= \ & (1+T)\log(w) \\
& + \mathbb{E}\left[\int_0^T (1+T-s)\left\{-\frac{1}{2}\boldsymbol{u}_s^\top \Sigma_s \Sigma_s^\top \boldsymbol{u}_s + \boldsymbol{u}_s^\top(\boldsymbol{\mu}_s - r_s \mathbf{1}_{n\times 1}) + r_s\right\} ds\right] \\
& + \mathbb{E}\left[\int_0^T \{\log(c_s) - (1+T-s)c_s\}\,ds\right].
\end{aligned} \tag{6.8}$$

Now it suffices to maximize the quadratic integrand

$$\boldsymbol{u} \mapsto -\frac{1}{2}\boldsymbol{u}^\top \Sigma_s \Sigma_s^\top \boldsymbol{u} + \boldsymbol{u}^\top(\boldsymbol{\mu}_s - r_s \mathbf{1}_{n\times 1}), \tag{6.9}$$

and the integrand $c \mapsto \{\log(c) - (1+T-t)c\}$. Finally elementary calculus will give us the optimal portfolio as of equation (6.2) and the optimal consumption as,

$$c_t = \frac{1}{1+T-t}. \tag{6.10}$$

Applying the direct approach is also straightforward. First we find the martingales $V^{(\boldsymbol{u})}$ and $V^{(\boldsymbol{u},c)}$:

$$\begin{aligned} V_t^{\boldsymbol{u}} &\triangleq \mathbb{E}_t\left[U'\left(W_T^{(\boldsymbol{u})}\right)W_T^{(\boldsymbol{u})}\right] = 1, \\ V_t^{(\boldsymbol{u},c)} &\triangleq \mathbb{E}_t\left[\int_t^T \frac{\partial}{\partial x}u\left(s, c_s W_s^{(\boldsymbol{u},c)}\right)c_s W_s^{(\boldsymbol{u},c)}ds + U'\left(W_T^{(\boldsymbol{u})}\right)W_T^{(\boldsymbol{u})}\right] = 1+T-t. \end{aligned} \tag{6.11}$$

Obviously $\left[\boldsymbol{W}, V^{(\boldsymbol{u})}\right]_t' = \left[\boldsymbol{W}, V^{(\boldsymbol{u},c)}\right]_t' = 0$. So for the case of zero consumption, by using equation (5.52), the optimal portfolio would be:

$$\boldsymbol{u}_t = \left(\Sigma_t\Sigma_t^T\right)^{-1}\left(\boldsymbol{\mu}_t - r_t\mathbf{1}_{n\times1}\right), \tag{6.12}$$

which is the same as what we found earlier by martingale approach. For the case of non-zero consumption, equation (5.55) will give use the same portfolio strategy as above, and the optimal consumption as:

$$V_t^{(\boldsymbol{u},c)} = \frac{\partial}{\partial x}u\left(t, c_t W_t^{(\boldsymbol{u},c)}\right)W_t^{(\boldsymbol{u},c)} \Longrightarrow c_t = \frac{1}{1+T-t}, \tag{6.13}$$

which is what we have already found in equation (6.10).

Finally let's take the martingale approach. Besides proving that the optimal portfolio and consumption is of the form (6.2) and (6.10), we will also prove that the optimal market price of risk, and the optimal wealth process are given by:

$$\boldsymbol{\lambda} = \Sigma^T\left(\Sigma\Sigma^T\right)^{-1}\boldsymbol{b}, \tag{6.14}$$

$$\begin{aligned} W_t^{(C^{(\boldsymbol{\lambda})},Z^{(\boldsymbol{\lambda})},\boldsymbol{\lambda})} &= \frac{w\left(1+T-t\right)}{(1+T)}e^{\int_0^t\left(r_s + (\boldsymbol{\mu}_s - r_s\mathbf{1}_{n\times1})^T\left(\Sigma_s\Sigma_s^T\right)^{-1}(\boldsymbol{\mu}_s - r_s\mathbf{1}_{n\times1})\right)ds} \\ &\quad\times\mathcal{E}\left(\int_0^{\cdot}\left(\boldsymbol{\mu} - r\mathbf{1}_{n\times1}\right)^T\left(\Sigma_s\Sigma_s^T\right)^{-1}\Sigma_s d\boldsymbol{W}_s\right)_t. \end{aligned} \tag{6.15}$$

We start by finding an expression for $\eta\left(\boldsymbol{\lambda}\right)$ of lemma 4.4. Note that $\frac{\partial}{\partial x}u\left(t,x\right) = U'\left(x\right) = \frac{1}{x}$, so $I(y,t) = I_F(y) = \frac{1}{y}$ and we obtain:

$$\mathbb{E}\left[\pi_T^{(\boldsymbol{\lambda})}I_F(\eta\left(\boldsymbol{\lambda}\right)\pi_T^{(\boldsymbol{\lambda})}) + \int_0^T \pi_\tau^{(\boldsymbol{\lambda})}I(\eta\left(\boldsymbol{\lambda}\right)\pi_\tau^{(\boldsymbol{\lambda})},\tau)d\tau\right] = \frac{1+T}{\eta\left(\boldsymbol{\lambda}\right)}. \tag{6.16}$$

Then equation (4.6) will give us $\eta\left(\boldsymbol{\lambda}\right) = \frac{1+T}{w}$, and we have:

$$C_t^{(\boldsymbol{\lambda})} = I(\eta\left(\boldsymbol{\lambda}\right)\pi_t^{(\boldsymbol{\lambda})},t) = \frac{1}{\eta\left(\boldsymbol{\lambda}\right)\pi_t^{(\boldsymbol{\lambda})}} = \frac{w}{(1+T)\,\pi_t^{(\boldsymbol{\lambda})}}, \tag{6.17}$$

$$Z^{(\boldsymbol{\lambda})} = I_F(\eta\left(\boldsymbol{\lambda}\right)\pi_T^{(\boldsymbol{\lambda})}) = \frac{1}{\eta\left(\boldsymbol{\lambda}\right)\pi_T^{(\boldsymbol{\lambda})}} = \frac{w}{(1+T)\,\pi_T^{(\boldsymbol{\lambda})}}. \tag{6.18}$$

The next step is to derive the process $\pi^{(\boldsymbol{\lambda})} W^{(C^{(\boldsymbol{\lambda})},Z^{(\boldsymbol{\lambda})},\boldsymbol{\lambda})}$. From equation (4.4) we may conclude:

$$\begin{aligned}\pi_t^{(\boldsymbol{\lambda})} W_t^{(C^{(\boldsymbol{\lambda})},Z^{(\boldsymbol{\lambda})},\boldsymbol{\lambda})} &= \mathbb{E}_t\left[\pi_T^{(\boldsymbol{\lambda})} Z^{(\boldsymbol{\lambda})} + \int_t^T \pi_\tau^{(\boldsymbol{\lambda})} C_\tau^{(\boldsymbol{\lambda})} d\tau\right] \\ &= \frac{w\,(1+T-t)}{(1+T)}. \qquad (6.19)\end{aligned}$$

But then $\left[\boldsymbol{W}, \pi^{(\boldsymbol{\lambda})} W^{(C,Z,\boldsymbol{\lambda})}\right]_t' = 0$, so $\boldsymbol{a}^{(C^{(\boldsymbol{\lambda})},Z^{(\boldsymbol{\lambda})},\boldsymbol{\lambda})} \triangleq \frac{\left[\boldsymbol{W},\pi^{(\boldsymbol{\lambda})} W^{(C,Z,\boldsymbol{\lambda})}\right]_t'}{\pi_t^{(\boldsymbol{\lambda})} W_t^{(C,Z,\boldsymbol{\lambda})}} = 0$ and equation (4.29) would give us $\boldsymbol{\lambda}$ as in equation (6.14). Now we may use theorem (4.9) to conclude that, only for this particular MPR $\boldsymbol{\lambda}$, $\left(C_t^{(\boldsymbol{\lambda})}, Z^{(\boldsymbol{\lambda})}\right)$ is the optimal consumption pair. To find equation (6.10), we observe that

$$c_t = \frac{C_t^{(\boldsymbol{\lambda})}}{W_t^{(C^{(\boldsymbol{\lambda})},Z^{(\boldsymbol{\lambda})},\boldsymbol{\lambda})}} = \frac{w}{(1+T)\,\pi_t^{(\boldsymbol{\lambda})} W_t^{(C^{(\boldsymbol{\lambda})},Z^{(\boldsymbol{\lambda})},\boldsymbol{\lambda})}}. \qquad (6.20)$$

Now by substituting for $\pi_t^{(\boldsymbol{\lambda})} W_t^{(C^{(\boldsymbol{\lambda})},Z^{(\boldsymbol{\lambda})},\boldsymbol{\lambda})}$ in the denominator from equation (6.19) we will obtain the result. Equation (6.2) directly follows from equation (4.30). Finally by substituting for $\pi_t^{(\boldsymbol{\lambda})}$ from equation (2.4) into equation (6.19) we would have:

$$\begin{aligned}W_t^{(C^{(\boldsymbol{\lambda})},Z^{(\boldsymbol{\lambda})},\boldsymbol{\lambda})} &= \frac{w\,(1+T-t)}{(1+T)\,\pi_t^{(\boldsymbol{\lambda})}} \\ &= \frac{w\,(1+T-t)}{(1+T)\,e^{-\int_0^t r_u du}\mathcal{E}\left(-\int_0^\cdot \boldsymbol{\lambda}_u^T d\boldsymbol{W}_u\right)} \\ &= \frac{w\,(1+T-t)}{(1+T)} e^{\int_0^t \left(r_s + \boldsymbol{\lambda}_s^T \boldsymbol{\lambda}_s\right)ds}\mathcal{E}\left(\int_0^\cdot \boldsymbol{\lambda}_s^T d\boldsymbol{W}_s\right)_t. \qquad (6.21)\end{aligned}$$

And substituting for $\boldsymbol{\lambda}$ from equation (6.14) will give us equation (6.15).

6.2 Power Utility

In this section we take the power utility functional, defined as:

$$J(C,Z) \triangleq \mathbb{E}\left[\int_0^T \frac{C_t^\gamma}{\gamma} dt + \frac{Z^\gamma}{\gamma}\right]. \qquad (6.22)$$

Then we consider the problem originally solved by Merton, and redrive the solution. After that we consider the case where the interest rate is stochastic, and follows a multi-factor Gaussian model. The optimal portfolio in the two cases differ in the following ways (compare theorem 6.3 and theorem 6.8),

- In the Merton setting, the optimal portfolio only includes the myopic term, while the excess hedging demand term (containing the covariance between zero-coupon bonds and assets) enters in the optimal portfolio for the Gaussian term structure case.

- In the Merton setting the optimal portfolio is deterministic whether we consider consumption or not, but for the Gaussian term structure case the optimal portfolio is deterministic when there is no consumption, and stochastic otherwise.

6.2.1 Original Merton Setting

Consider a market with deterministic coefficients, defined as follow.

Definition 6.1. A *market with deterministic coefficients* is defined as a special case of definition 2.1 where r, $\boldsymbol{\mu}$, and Σ are deterministic functions of time, and $n = k$ (i.e. the market is complete). In this case we write:

$$\begin{aligned} r_t &= r(t) \\ \boldsymbol{\mu}_t &= \boldsymbol{\mu}(t) \\ \Sigma_t &= \Sigma(t). \end{aligned} \tag{6.23}$$

After deriving a useful result in lemma 6.2, we will identify Merton's solution in theorem 6.3.

Lemma 6.2. *Consider the market defined in definition6.1 and let $\boldsymbol{\lambda}(t)$ and $\pi = (\pi_t)$ be the unique market-price-of-risk and SPD identified in theorem 2.3, equations (2.8) and (2.4), respectively . Then for any constant ξ we have:*

$$\pi_t^{\xi} = m(t) \times \Lambda_t \qquad \text{for all } t, \tag{6.24}$$

$$\mathbb{E}_s\left[\pi_t^{\xi}\right] = m(t) \times \Lambda_s \qquad \text{for all } s < t, \tag{6.25}$$

where the deterministic function $m(t)$ and the martingale $\Lambda = (\Lambda_t)$ are defined as:

$$\begin{aligned} m(t) &\triangleq e^{\xi \int_0^t \left\{-r(s)+\frac{\xi-1}{2}\boldsymbol{\lambda}^T(s)\boldsymbol{\lambda}(s)\right\}ds}, \\ \Lambda &\triangleq \mathcal{E}\left(-\xi \int_0^{\cdot} \boldsymbol{\lambda}^T(s)\, d\boldsymbol{W}_s\right). \end{aligned} \tag{6.26}$$

Proof. By using proposition A.4 (part iii) of appendix A, we obtain equation (6.24) as follow:

$$\begin{aligned} \pi_t^{\xi} &= \left(e^{-\int_0^t r_u du}\mathcal{E}\left(-\int_0^{\cdot} \boldsymbol{\lambda}^T(s)\, d\boldsymbol{W}_s\right)_t\right)^{\xi} \\ &= e^{-\xi\int_0^t r(s)ds}\left(\mathcal{E}\left(-\int_0^{\cdot} \boldsymbol{\lambda}^T(s)\, d\boldsymbol{W}_s\right)_t\right)^{\xi} \\ &= e^{-\xi\int_0^t r(s)ds}\left(e^{\frac{\xi(\xi-1)}{2}[-\int_0^{\cdot}\boldsymbol{\lambda}^T(s)d\boldsymbol{W}_s]_t}\mathcal{E}\left(-\xi\int_0^{\cdot} \boldsymbol{\lambda}^T(s)\, d\boldsymbol{W}_s\right)_t\right) \\ &= e^{\xi \int_0^t \left\{-r(s)+\frac{\xi-1}{2}\boldsymbol{\lambda}^T(s)\boldsymbol{\lambda}(s)\right\}ds}\mathcal{E}\left(-\xi\int_0^{\cdot} \boldsymbol{\lambda}^T(s)\, d\boldsymbol{W}_s\right)_t. \end{aligned} \tag{6.27}$$

To obtain equation (6.25), we only need to take conditional expectation from the last result, and use the fact that $m(.)$ is deterministic and Λ is a martingale:

$$\begin{aligned} \mathbb{E}_s\left[\pi_t^{\xi}\right] &= \mathbb{E}_s\left[m(t) \times \Lambda_t\right] \\ &= m(t)\mathbb{E}_s\left[\Lambda_t\right] \\ &= m(t) \times \Lambda_s. \end{aligned} \tag{6.28}$$

□

Theorem 6.3. *Consider the market defined in definition6.1 . Then:*

(i) Consider the Merton's problem with the total utility as:

$$J(u) = \mathbb{E}\left[\frac{\left(W_T^{(u)}\right)^\gamma}{\gamma}\right], \tag{6.29}$$

where $\gamma > 0$ *is a constant. Then the optimal portfolio is deterministic and is given by:*

$$\boldsymbol{u} = \frac{1}{1-\gamma}\left(\Sigma\Sigma^T\right)^{-1}(\boldsymbol{\mu} - r\mathbf{1}_{n\times 1}) \tag{6.30}$$

(ii) Consider the Merton's problem with the total utility as:

$$J(c,u) = \mathbb{E}\left[\int_0^T \frac{\left(c_t W_t^{(u,c)}\right)^\gamma}{\gamma} dt + \frac{\left(W_T^{(u,c)}\right)^\gamma}{\gamma}\right]$$

i.e. the case of power utility with consumption. Then the optimal portfolio is the same as the case of no consumption (i.e. equation 6.30). The optimal proportional consumption is also deterministic and is given by:

$$c_t = \frac{m(t)}{\int_t^T m(s)ds + m(T)}, \tag{6.31}$$

where the function $m(t)$ *is defined as:*

$$m(t) \triangleq e^{\xi \int_0^t \left\{-r(s) + \frac{\xi-1}{2}\boldsymbol{\lambda}^T(s)\boldsymbol{\lambda}(s)\right\}ds}, \tag{6.32}$$

with $\xi = \frac{\gamma}{\gamma-1}$.

Proof. As pointed out in remark 4.10, since the market is complete, equation (4.29) holds automatically. We only need to find an expression for $\boldsymbol{a}^{\left(C^{(\boldsymbol{\lambda})}, Z^{(\boldsymbol{\lambda})}, \boldsymbol{\lambda}\right)}$, and then equations (4.27) and (4.35) will give us the optimal solution. Also note that since the market is complete and the unique MPR is $\boldsymbol{\lambda} = \Sigma^{-1}(\boldsymbol{\mu} - r\mathbf{1}_{n\times 1})$, we may drop the function argument $(\boldsymbol{\lambda})$ in $\eta(\boldsymbol{\lambda})$, $\pi^{(\boldsymbol{\lambda})}$,etc.

(i) To apply the martingale approach we start with the function η of lemma 4.4. We have $I_F(y) = y^{\frac{1}{\gamma-1}}$. So η can be derived as follow:

$$\begin{aligned} \mathbb{E}\left[\pi_T I_F(\eta\pi_T)\right] &= \eta^{\frac{1}{\gamma-1}}\mathbb{E}\left[\pi_T^{\frac{\gamma}{\gamma-1}}\right] = w \\ &\Longrightarrow \eta = \left(\frac{w}{\mathbb{E}\left[\pi_T^{\frac{\gamma}{\gamma-1}}\right]}\right)^{\gamma-1}. \end{aligned} \tag{6.33}$$

The next goal is to find an expression for $\boldsymbol{a}^{\left(0, Z^{(\boldsymbol{\lambda})}, \boldsymbol{\lambda}\right)}$ defined in theorem 4.9. Note that by taking $\xi \triangleq \frac{\gamma}{\gamma-1}$ in lemma 6.2 we have $\mathbb{E}_t\left[\pi_T^\xi\right] = m(T) \times \Lambda_t$, from which we may also conclude:

$$\mathbb{E}\left[\pi_T^\xi\right] = \mathbb{E}_0\left[\pi_T^\xi\right] = m(T) \times \Lambda_0 = m(T). \tag{6.34}$$

Now consider the martingale $\pi^{(\boldsymbol{\lambda})}W^{(0,Z^{(\boldsymbol{\lambda})},\boldsymbol{\lambda})}$ of equation (4.4),

$$\begin{aligned}
\pi_t^{(\boldsymbol{\lambda})}W_t^{(0,Z^{(\boldsymbol{\lambda})},\boldsymbol{\lambda})} &\triangleq \mathbb{E}_t\left[\pi_T I_F(\eta\pi_T)\right] \\
&= \eta^{\frac{1}{\gamma-1}}\mathbb{E}_t\left[\pi_T^{\frac{\gamma}{\gamma-1}}\right] \\
&= \left(\frac{w}{m(T)}\right)(m(T)\times\Lambda_t) \\
&= w\times\Lambda_t \\
&= w\mathcal{E}\left(-\frac{\gamma}{\gamma-1}\int_0^{\cdot}\boldsymbol{\lambda}^T(s)\,d\boldsymbol{W}_s\right)_t.
\end{aligned} \tag{6.35}$$

This last result gives us

$$\begin{aligned}
\boldsymbol{a}^{\left(0,Z^{(\boldsymbol{\lambda})},\boldsymbol{\lambda}\right)} &\triangleq \frac{\left[\boldsymbol{W},\pi^{(\boldsymbol{\lambda})}W^{(C,Z,\boldsymbol{\lambda})}\right]_t'}{\pi_t^{(\boldsymbol{\lambda})}W_t^{(0,Z^{(\boldsymbol{\lambda})},\boldsymbol{\lambda})}} \\
&= -\frac{\gamma}{\gamma-1}\boldsymbol{\lambda}(t) \\
&= -\frac{\gamma}{\gamma-1}\Sigma^{-1}\boldsymbol{b}.
\end{aligned} \tag{6.36}$$

Finally by using equation (4.35) we find the optimal portfolio:

$$\begin{aligned}
\boldsymbol{u} &= \left(\Sigma\Sigma^T\right)^{-1}\boldsymbol{b}+\left(\Sigma^T\right)^{-1}\boldsymbol{a}^{(C^{(\boldsymbol{\lambda})},Z^{(\boldsymbol{\lambda})},\boldsymbol{\lambda})} \\
&= \left(\Sigma\Sigma^T\right)^{-1}\boldsymbol{b}-\left(\Sigma^T\right)^{-1}\frac{\gamma}{\gamma-1}\Sigma^{-1}\boldsymbol{b} \\
&= \frac{1}{1-\gamma}\left(\Sigma\Sigma^T\right)^{-1}\boldsymbol{b}.
\end{aligned} \tag{6.37}$$

(ii) Here $u(x,t)=U(x)=\frac{x^\gamma}{\gamma}$, so $I(y,t)=I_F(y)=y^{\frac{1}{\gamma-1}}$. Hence equation 4.6 would become:

$$\begin{aligned}
w &= \mathbb{E}\left[\int_0^T \pi_t I(\eta\pi_t,t)dt+\pi_T I_F(\eta\pi_T)\right] \\
&= \mathbb{E}\left[\int_0^T \pi_t\left(\eta\pi_t\right)^{\frac{1}{\gamma-1}}dt+\pi_T\left(\eta\pi_T\right)^{\frac{1}{\gamma-1}}\right] \\
&= \eta^{\frac{1}{\gamma-1}}\mathbb{E}\left[\int_0^T \pi_t^{\frac{\gamma}{\gamma-1}}dt+\pi_T^{\frac{\gamma}{\gamma-1}}\right],
\end{aligned} \tag{6.38}$$

And we obtain η as:

$$\eta=\left(\frac{w}{\mathbb{E}\left[\int_0^T\pi_t^{\frac{\gamma}{\gamma-1}}dt+\pi_T^{\frac{\gamma}{\gamma-1}}\right]}\right)^{\gamma-1} \tag{6.39}$$

To find an expression for $\boldsymbol{a}^{\left(C^{(\boldsymbol{\lambda})},Z^{(\boldsymbol{\lambda})},\boldsymbol{\lambda}\right)}$, first note that from equation 6.25 of lemma 6.2, we have $\mathbb{E}_t\left[\pi_s^\xi\right] = m(s)\times\Lambda_t$ for all $t<s$. So:

$$\begin{aligned}
\mathbb{E}_t\left[\int_t^T \pi_s^\xi ds + \pi_T^\xi\right] &= \int_t^T \mathbb{E}_t\left[\pi_s^\xi\right] ds + \mathbb{E}_t\left[\pi_T^\xi\right] \\
&= \int_t^T m(s)\times\Lambda_t ds + m(T)\times\Lambda_t \\
&= \left(\int_t^T m(s)ds + m(T)\right)\times\Lambda_t. \qquad (6.40)
\end{aligned}$$

Now by applying the product rule we obtain:

$$\begin{aligned}
\mathbb{E}_t\left[\int_t^T \pi_s^\xi ds + \pi_T^\xi\right] &= \int_0^T m(s)ds + m(T) \\
&\quad + \int_0^t \Lambda_\tau d\left(\int_.^T m(s)ds + m(T)\right)_\tau \\
&\quad + \int_0^t \left(\int_\tau^T m(s)ds + m(T)\right) d\Lambda_\tau \\
&= \int_0^T m(s)ds + m(T) - \int_0^t \Lambda_\tau m(\tau)\, d\tau \\
&\quad - \int_0^t \left(\int_\tau^T m(s)ds + m(T)\right)\Lambda_\tau \xi \boldsymbol{\lambda}^T(\tau)\, d\boldsymbol{W}_\tau \\
&= \int_0^T m(s)ds + m(T) - \int_0^t \Lambda_\tau m(\tau)\, d\tau \\
&\quad - \int_0^t \mathbb{E}_\tau\left[\int_\tau^T \pi_s^\xi ds + \pi_T^\xi\right]\xi\boldsymbol{\lambda}^T(\tau)\, d\boldsymbol{W}_\tau. \qquad (6.41)
\end{aligned}$$

In the last step we used equation (6.40). From this last result we may conclude:

$$\mathbb{E}\left[\int_0^T \pi_s^\xi ds + \pi_T^\xi\right] = \int_0^T m(s)ds + m(T) \qquad (6.42)$$

$$\frac{\left[\boldsymbol{W}, \mathbb{E}_.\left[\int_.^T \pi_s^\xi ds + \pi_T^\xi\right]\right]_t'}{\mathbb{E}_t\left[\int_t^T \pi_s^\xi ds + \pi_T^\xi\right]} = -\xi\boldsymbol{\lambda}(\tau) \qquad (6.43)$$

From the first equation above we will find the following explicit form for η:

$$\eta^{\frac{1}{\gamma-1}} = \frac{w}{\mathbb{E}\left[\int_0^T \pi_s^\xi ds + \pi_T^\xi\right]} = \frac{w}{\int_0^T m(s)ds + m(T)}. \qquad (6.44)$$

Now consider the process $\pi^{(\boldsymbol{\lambda})}W^{(C^{(\boldsymbol{\lambda})},Z^{(\boldsymbol{\lambda})},\boldsymbol{\lambda})}$ of equation (4.4):

$$\begin{aligned}
\pi_t^{(\boldsymbol{\lambda})}W_t^{\left(C^{(\boldsymbol{\lambda})},Z^{(\boldsymbol{\lambda})},\boldsymbol{\lambda}\right)} &\triangleq \mathbb{E}_t\left[\int_t^T \pi_t I(\eta\pi_t, t)dt + \pi_T I_F(\eta\pi_T)\right] \\
&= \eta^{\frac{1}{\gamma-1}}\mathbb{E}_t\left[\int_t^T \pi_s^\xi ds + \pi_T^\xi\right], \qquad (6.45)
\end{aligned}$$

where we took $\xi = \frac{\gamma}{\gamma-1}$. By using equation (6.43) we may conclude

$$\begin{aligned}
\boldsymbol{a}_t^{\left(C^{(\boldsymbol{\lambda})},Z^{(\boldsymbol{\lambda})},\boldsymbol{\lambda}\right)} &\triangleq \frac{\left[\boldsymbol{W}, \pi^{(\boldsymbol{\lambda})} W^{\left(C^{(\boldsymbol{\lambda})},Z^{(\boldsymbol{\lambda})},\boldsymbol{\lambda}\right)}\right]_t'}{\pi_t^{(\boldsymbol{\lambda})} W_t^{\left(C^{(\boldsymbol{\lambda})},Z^{(\boldsymbol{\lambda})},\boldsymbol{\lambda}\right)}} \\
&= \frac{\left[\boldsymbol{W}, \mathbb{E}_t\left[\int_t^T \pi_s^{\xi} ds + \pi_T^{\xi}\right]\right]_t'}{\mathbb{E}_t\left[\int_t^T \pi_s^{\xi} ds + \pi_T^{\xi}\right]} = -\xi \boldsymbol{\lambda}(\tau)
\end{aligned}$$

But this is exactly what we found in equation (6.36) for the case of zero consumption. Hence the optimal portfolio would be also given by equation (6.30). To obtain the optimal consumption, we may proceed as follow:

$$\begin{aligned}
\pi_t C_t &= \pi_t I(\eta \pi_t, t) \\
&= \eta^{\frac{1}{\gamma-1}} \pi_t^{\xi} \\
&= \eta^{\frac{1}{\gamma-1}} \times m(t) \times \Lambda_t \\
&= \frac{m(t)}{\int_t^T m(s)ds + m(T)} \eta^{\frac{1}{\gamma-1}} \left(\int_t^T m(s)ds + m(T)\right) \Lambda_t \\
&= \frac{m(t)}{\int_t^T m(s)ds + m(T)} \eta^{\frac{1}{\gamma-1}} \mathbb{E}_t\left[\int_t^T \pi_s^{\xi} ds + \pi_T^{\xi}\right] \\
&= \frac{m(t)}{\int_t^T m(s)ds + m(T)} \pi_t W_t^{\left(C^{(\boldsymbol{\lambda})},Z^{(\boldsymbol{\lambda})},\boldsymbol{\lambda}\right)}. \qquad (6.46)
\end{aligned}$$

Finally by using the relation $c_t = \frac{C_t}{W_t^{\left(C^{(\boldsymbol{\lambda})},Z^{(\boldsymbol{\lambda})},\boldsymbol{\lambda}\right)}}$, we will get the optimal proportional consumption of equation (6.31). □

6.2.2 Complete Gaussian Market

Consider the complete Gaussian market defined bellow.

Definition 6.4. By a *Gaussian Market* we are referring to the special case of definition 2.1 where the volatility and the market price of risk process are deterministic, i.e. $\Sigma_t = \Sigma(t)$ and $\boldsymbol{\lambda}_t = \boldsymbol{\lambda}(t)$. Furthermore the short-rate process r follows a k-factor Gaussian short rate model. That is, we have:

$$r_t = \sum_{i=1}^{k} r_t^{(i)} = \mathbf{1}_{1\times k} \boldsymbol{r}_t, \qquad (6.47)$$

where $\boldsymbol{r} = \left(r^{(1)}, \dots, r^{(k)}\right)$ are independent Gaussian process with the following dynamics:

$$dr_t^{(i)} = \left(\vartheta^{(i)}(t) + \alpha^{(i)}(t)\, r_t^{(i)}\right) dt + \delta^{(i)}(t) d\widetilde{\mathcal{W}}_t^{(i)} \qquad for\, i = 1, \dots, k, \qquad (6.48)$$

or equivalently,

$$d\boldsymbol{r}_t = \left(\boldsymbol{\theta}(t) + \mathrm{diag}\left(\boldsymbol{\alpha}(t)\right) \boldsymbol{r}_t\right) dt + \mathrm{diag}\left(\boldsymbol{\delta}(t)\right) d\widetilde{\boldsymbol{W}}_t. \qquad (6.49)$$

Here $\vartheta^{(i)}(.)$, $\alpha^{(i)}(.)$ and $\delta^{(i)}(.)$ are given deterministic functions.

We will also need to consider τ-maturity bonds explicitly.

Definition 6.5. The τ-maturity bond price process $B^{(\tau)} = \left(B_t^{(\tau)}\right)$ is defined as:

$$B_t^{(\tau)} \triangleq \mathbb{E}_t^{\mathbb{Q}}\left(e^{-\int_t^\tau r_s ds}\right) \tag{6.50}$$

Also assume $\boldsymbol{\sigma}^{(\tau)}(t)$ to be the *deterministic* volatility of $B^{(\tau)}$ and $\boldsymbol{\rho}^{(\boldsymbol{A},\tau)} = \left(\rho^{(1,\tau)}(t), \ldots, \rho^{(n,\tau)}(t)\right)^T$ to be the *deterministic* instantaneous correlation between the returns of assets $\boldsymbol{A}$ and the τ-maturity bond $B^{(\tau)}$.

The following lemma will introduce a family of martingales $F^{(\tau)} = \left(F_t^{(\tau)}\right)$ which play a main role in finding the optimal solutions.

Lemma 6.6. *For a maturity $\tau > 0$ and an arbitrary constant ξ, define a family of martingales $F^{(\tau)} = \left(F_t^{(\tau)}\right)$ as:*

$$\mathcal{E}\left(F^{(\tau)}\right)_t \triangleq \mathbb{E}_t\left(e^{-\xi\int_0^\tau r_s ds}\right) \tag{6.51}$$

Then for the Brownian motion $\boldsymbol{W}$ we have:

$$\left[\boldsymbol{W}, F^{(\tau)}\right]' = \xi\boldsymbol{\sigma}^{(\tau)}. \tag{6.52}$$

Proof. First we show that for some deterministic function $f(\tau)$ we have:

$$\mathbb{E}_t\left(e^{-\xi\int_0^\tau r_u du}\right) = f(\tau)\mathbb{E}_t^{\mathbb{Q}}\left(e^{-\xi\int_0^\tau r_u du}\right). \tag{6.53}$$

To see this, recall from equation (2.5) that $d\widetilde{\boldsymbol{W}}_u = d\boldsymbol{\mathcal{W}}_u + \boldsymbol{\lambda}(u)\, du$. Now from equation(B.2) of appendix B, we have:

$$\begin{aligned}
\boldsymbol{r}_t &= \Upsilon(t)\left\{\boldsymbol{r}_0 + \int_0^t \Upsilon^{-1}(u)\boldsymbol{\theta}(u)du + \int_0^t \Upsilon^{-1}(u)\mathrm{diag}\left(\boldsymbol{\delta}(t)\right)d\widetilde{\boldsymbol{W}}_u\right\} \\
&= \Upsilon(t)\left\{\boldsymbol{r}_0 + \int_0^t \Upsilon^{-1}(u)\boldsymbol{\theta}(u)du + \int_0^t \Upsilon^{-1}(u)\mathrm{diag}\left(\boldsymbol{\delta}(t)\right)d\boldsymbol{\mathcal{W}}_u\right\} \\
&\quad +\Upsilon(t)\int_0^t \Upsilon^{-1}(u)\mathrm{diag}\left(\boldsymbol{\delta}(t)\right)\boldsymbol{\lambda}(u)du \\
&\triangleq \tilde{\boldsymbol{r}}_t + \boldsymbol{g}(t).
\end{aligned} \tag{6.54}$$

Define $\tilde{r} \triangleq \mathbf{1}_{1\times k}\tilde{\boldsymbol{r}}$ and $g(t) \triangleq \mathbf{1}_{1\times k}\boldsymbol{g}(t)$. Note that $r_t = \tilde{r}_t + g(t)$. Also, since the $\mathbb{Q}$-dynamics of $\boldsymbol{r}$ and $\mathbb{P}$-dynamics of $\tilde{\boldsymbol{r}}$ are identical, the same is true for r and $\tilde{r}$. So:

$$\begin{aligned}
\mathbb{E}_t\left(e^{-\xi\int_0^\tau r_u du}\right) &= e^{-\xi\int_0^\tau g(u)du}\mathbb{E}_t\left(e^{-\xi\int_0^\tau \tilde{r}_u du}\right) \\
&= e^{-\xi\int_0^\tau g(u)du}\mathbb{E}_t^{\mathbb{Q}}\left(e^{-\xi\int_0^\tau r_u du}\right).
\end{aligned} \tag{6.55}$$

And we may take $f(\tau) \triangleq e^{-\xi\int_0^\tau g(u)du}$.

Now it is a well known fact that $\int_s^t r_u du|\mathcal{F}_s$ is Gaussian, and that its conditional variance is deterministic (to see this in a special case, refer to equation B.7 in appendix B). So we may write:

$$\begin{aligned}
\mathcal{E}\left(F^{(\tau)}\right)_t &\triangleq \mathbb{E}_t\left(e^{-\xi\int_0^\tau r_u du}\right) \\
&= f(\tau)\mathbb{E}_t^Q\left(e^{-\xi\int_0^\tau r_u du}\right) \\
&= f(\tau)e^{-\xi\int_0^t r_u du}\mathbb{E}_t^Q\left(e^{-\xi\int_t^\tau r_u du}\right) \\
&= f(\tau)e^{-\xi\int_0^t r_u du}e^{-\xi\mathbb{E}_t^Q\left(\int_t^\tau r_u du\right)+\frac{\xi^2}{2}\mathrm{Var}_t\left(\int_t^\tau r_u du\right)} \\
&= f(\tau)e^{-\xi\int_0^t r_u du}e^{\left(\frac{\xi^2-\xi}{2}\right)\mathrm{Var}_t\left(\int_t^\tau r_u du\right)}\left(e^{-\mathbb{E}_t^{\mathbb{Q}}\left(\int_t^\tau r_u du\right)+\frac{1}{2}\mathrm{Var}_t\left(\int_t^\tau r_u du\right)}\right)^\xi \\
&= f(\tau)e^{-\xi\int_0^t r_u du}e^{\left(\frac{\xi^2-\xi}{2}\right)\mathrm{Var}_t\left(\int_t^\tau r_u du\right)}\left(\mathbb{E}_t^{\mathbb{Q}}\left(e^{-\int_t^\tau r_u du}\right)\right)^\xi \\
&= f(\tau)e^{-\xi\int_0^t r_u du}e^{\left(\frac{\xi^2-\xi}{2}\right)\mathrm{Var}_t\left(\int_t^\tau r_u du\right)}\left(B_t^{(\tau)}\right)^\xi. \qquad (6.56)
\end{aligned}$$

Since $\mathrm{Var}_t\left(\int_t^\tau r_u du\right)$ is deterministic, we may conclude that:

$$F^{(\tau)} = \xi\mathcal{L}\mathrm{og}\left(B^{(\tau)}\right) + BV. \qquad (6.57)$$

for some process BV of bounded variation. From this we may conclude

$$\left[\boldsymbol{W}, F^{(\tau)}\right]_t' = \xi\left[\boldsymbol{W}, \mathcal{L}\mathrm{og}\left(B^{(\tau)}\right)\right]_t' = \xi\boldsymbol{\sigma}^{(\tau)}(t), \qquad (6.58)$$

which is equation (6.52). □

The following lemma is a counterpart of lemma 6.2, and will play a crucial rule.

Lemma 6.7. *Consider the market defined in definition 6.4 and let $\pi = (\pi_t)$ be the SPD process. Then for any constant ξ we have:*

$$\begin{aligned}
\pi_t^\xi &= m(t) \times \Lambda_t^{(t)}, \qquad (6.59) \\
\mathbb{E}_s\left[\pi_t^\xi\right] &= m(t)\Lambda_s^{(t)} \quad for\, s < t, \qquad (6.60)
\end{aligned}$$

where the deterministic function $m(t)$ and the family of martingales $\Lambda^{(t)}$ are defined as follow:

$$\begin{aligned}
m(t) &\triangleq e^{\xi\int_0^t\left(-\xi\boldsymbol{\sigma}^{(t)}(s)+\frac{\xi-1}{2}\boldsymbol{\lambda}(s)\right)^T\boldsymbol{\lambda}(s)ds}, \\
\Lambda^{(t)} &\triangleq \mathcal{E}\left(\xi\int_0^\cdot\left(\boldsymbol{\sigma}^{(t)}(s)-\boldsymbol{\lambda}(s)\right)^T d\boldsymbol{W}_s\right).
\end{aligned} \qquad (6.61)$$

Proof. With a calculation similar to the way we obtain equation(6.27), we have:

$$\begin{aligned}
\pi_t^\xi &= \left(e^{-\int_0^t r_s ds}\mathcal{E}\left(-\int_0^\cdot \boldsymbol{\lambda}^T(s)\, d\boldsymbol{W}_s\right)_t\right)^\xi \\
&= e^{\frac{\xi(\xi-1)}{2}\int_0^t\boldsymbol{\lambda}^T(s)\boldsymbol{\lambda}(s)ds}e^{-\xi\int_0^t r_s ds}\mathcal{E}\left(-\xi\int_0^\cdot \boldsymbol{\lambda}^T(s)\, d\boldsymbol{W}_s\right)_t. \qquad (6.62)
\end{aligned}$$

From equation(6.51) we have $\mathcal{E}\left(F^{(t)}\right)_t = e^{-\xi\int_0^t r_s ds}$. Now we can obtain equation(6.59) as follow:

$$
\begin{aligned}
\pi_t^{\xi} &= e^{\frac{\xi(\xi-1)}{2}\int_0^t \boldsymbol{\lambda}^T(s)\boldsymbol{\lambda}(s)ds}\mathcal{E}\left(F^{(t)}\right)_t \mathcal{E}\left(-\xi\int_0^{\cdot}\boldsymbol{\lambda}^T(s)\,d\boldsymbol{W}_s\right)_t \\
&= e^{\frac{\xi(\xi-1)}{2}\int_0^t \boldsymbol{\lambda}^T(s)\boldsymbol{\lambda}(s)ds} e^{[F^{(t)},-\xi\int_0^{\cdot}\boldsymbol{\lambda}^T(s)d\boldsymbol{\mathcal{W}}_s]_t}\mathcal{E}\left(F^{(t)}-\xi\int_0^{\cdot}\boldsymbol{\lambda}^T(s)\,d\boldsymbol{W}_s\right)_t \\
&= e^{\xi\int_0^t\left\{-\boldsymbol{\lambda}^T(s)\left[\boldsymbol{\mathcal{W}},F^{(t)}\right]_s' + \frac{\xi-1}{2}\boldsymbol{\lambda}^T(s)\boldsymbol{\lambda}(s)\right\}ds}\mathcal{E}\left(\int_0^{\cdot}\left\{\left[F^{(t)},\boldsymbol{W}\right]_s' - \xi\boldsymbol{\lambda}^T(s)\right\}d\boldsymbol{W}_s\right)_t \\
&= m(t)\times\Lambda_t^{(t)},
\end{aligned}
\tag{6.63}
$$

where in the last steps we used equation (6.52). Finally equation (6.60) can be obtained by taking conditional expectation from equation(6.59). □

Finally the following theorem will give us the optimal solutions in this case.

Theorem 6.8. *Consider the market defined in definition2.1 special case (i). Then:*
(i) Consider the Merton's problem with the total utility as:

$$J(u) = \mathbb{E}\left[\frac{\left(W_T^{(u)}\right)^{\gamma}}{\gamma}\right]$$

i.e. the case of power utility with no consumption. Then the optimal portfolio is given by:

$$\boldsymbol{u}_t = \frac{1}{1-\gamma}\left(\Sigma_t^T\right)^{-1}\left(\Sigma_t^{-1}\boldsymbol{b}_t - \gamma\boldsymbol{\sigma}_t^{(T)}\right) \tag{6.64}$$

(ii) Consider the Merton's problem with the total utility as:

$$J(c,u) = \mathbb{E}\left[\int_0^T \frac{\left(c_t W_t^{(u,c)}\right)^{\gamma}}{\gamma}dt + \frac{\left(W_T^{(u,c)}\right)^{\gamma}}{\gamma}\right]$$

i.e. the case of power utility with consumption. Then the optimal portfolio and optimal consumption are given by:

$$
\begin{aligned}
\boldsymbol{u} &= \frac{1}{1-\gamma}\left(\Sigma\Sigma^T\right)^{-1}\boldsymbol{b} \\
&\quad -\frac{\gamma}{1-\gamma}\left(\Sigma^T\right)^{-1}\left\{\frac{\int_t^T m(s)\Lambda_t^{(s)}\boldsymbol{\sigma}^{(s)}(t)\,ds + m(T)\Lambda_t^{(T)}\boldsymbol{\sigma}^{(T)}(t)}{\int_t^T m(s)\Lambda_t^{(s)}ds + m(T)\Lambda_t^{(T)}}\right\},
\end{aligned}
\tag{6.65}
$$

$$c_t = \frac{m(t)\times\Lambda_t^{(t)}}{\int_t^T m(s)\times\Lambda_t^{(s)}ds + m(T)\times\Lambda_t^{(T)}}. \tag{6.66}$$

where the $m(t)$ and $\Lambda_t^{(\tau)}$ are defined in equation (6.61) with $\xi = \frac{\gamma}{\gamma-1}$.

Proof. (i) Note that by the same reasoning as in the proof of part(i) of theorem(6.3), equations (6.33) and (6.35), we will obtain:

$$\eta = \left(\frac{w}{\mathbb{E}\left[\pi_T^{\xi}\right]} \right)^{\gamma-1}, \tag{6.67}$$

$$\pi_t^{(\boldsymbol{\lambda})} W_t^{(0,Z^{(\boldsymbol{\lambda})},\boldsymbol{\lambda})} = w \Lambda_t^{(T)}, \tag{6.68}$$

where $\xi = \frac{\gamma}{\gamma-1}$. By using the definition of $\Lambda_t^{(T)}$, equation (6.61), we have:

$$\begin{aligned} \pi_t^{(\boldsymbol{\lambda})} W_t^{(0,Z^{(\boldsymbol{\lambda})},\boldsymbol{\lambda})} &= w \left(1 + \int_0^t \xi \Lambda_s^{(T)} \left(\boldsymbol{\sigma}^{(T)}(s) - \boldsymbol{\lambda}(s) \right)^T d\boldsymbol{W}_s \right) \\ &= w + \int_0^t \xi \pi_s^{(\boldsymbol{\lambda})} W_s^{(0,Z^{(\boldsymbol{\lambda})},\boldsymbol{\lambda})} \left(\boldsymbol{\sigma}^{(T)}(s) - \boldsymbol{\lambda}(s) \right)^T d\boldsymbol{W}_s. \end{aligned} \tag{6.69}$$

And we obtain

$$\boldsymbol{a}_t^{\left(0,Z^{(\boldsymbol{\lambda})},\boldsymbol{\lambda}\right)} \triangleq \frac{\left[\boldsymbol{W}, \pi^{(\boldsymbol{\lambda})} W^{(C,Z,\boldsymbol{\lambda})} \right]_t'}{\pi_t^{(\boldsymbol{\lambda})} W_t^{(0,Z^{(\boldsymbol{\lambda})},\boldsymbol{\lambda})}} = \xi \left(\boldsymbol{\sigma}^{(T)}(t) - \boldsymbol{\lambda}(t) \right). \tag{6.70}$$

Finally by using equation (4.35) we find the optimal portfolio:

$$\begin{aligned} \boldsymbol{u} &= \left(\Sigma \Sigma^T \right)^{-1} \boldsymbol{b} + \left(\Sigma^T \right)^{-1} \boldsymbol{a}^{(C^{(\boldsymbol{\lambda})}, Z^{(\boldsymbol{\lambda})}, \boldsymbol{\lambda})} \\ &= \left(\Sigma \Sigma^T \right)^{-1} \boldsymbol{b} + \xi \left(\Sigma^T \right)^{-1} \left(\boldsymbol{\sigma}^{(T)} - \Sigma^{-1} \boldsymbol{b} \right) \\ &= (1-\xi) \left(\Sigma \Sigma^T \right)^{-1} \boldsymbol{b} + \xi \left(\Sigma^T \right)^{-1} \boldsymbol{\sigma}^{(T)}. \\ &= \frac{1}{1-\gamma} \left(\Sigma^T \right)^{-1} \left(\Sigma^{-1} \boldsymbol{b} - \gamma \boldsymbol{\sigma}^{(T)} \right). \end{aligned} \tag{6.71}$$

(ii) By a similar argument as what we used to obtain equations (6.39) and (6.45), we will have:

$$\eta = \left(\frac{w}{\mathbb{E}\left[\int_0^T \pi_t^{\frac{\gamma}{\gamma-1}} dt + \pi_T^{\frac{\gamma}{\gamma-1}} \right]} \right)^{\gamma-1}, \tag{6.72}$$

$$\pi_t^{(\boldsymbol{\lambda})} W_t^{\left(C^{(\boldsymbol{\lambda})},Z^{(\boldsymbol{\lambda})},\boldsymbol{\lambda}\right)} = \eta^{\frac{1}{\gamma-1}} \mathbb{E}_t \left[\int_t^T \pi_s^{\xi} ds + \pi_T^{\xi} \right]. \tag{6.73}$$

Also by using lemma (6.7) we have:

$$\begin{aligned} \mathbb{E}_t \left[\int_t^T \pi_s^{\xi} ds + \pi_T^{\xi} \right] &= \int_t^T \mathbb{E}_t \left[\pi_s^{\xi} \right] ds + \mathbb{E}_t \left[\pi_T^{\xi} \right] \\ &= \int_t^T m(s) \Lambda_t^{(s)} ds + m(T) \Lambda_t^{(T)}. \end{aligned} \tag{6.74}$$

Note that:

$$\begin{aligned} d \left(\int_{\cdot}^T m(s) \times \Lambda_{\cdot}^{(s)} ds \right)_\tau &= \int_\tau^T m(s) \times d\left(\Lambda_\tau^{(s)} \right) ds - m(\tau) \times \Lambda_\tau^{(t)} d\tau \\ &= \left(\int_\tau^T \xi m(s) \Lambda_\tau^{(s)} \left(\boldsymbol{\sigma}^{(s)}(\tau) - \boldsymbol{\lambda}(\tau) \right)^T ds \right) d\boldsymbol{W}_\tau \\ &\quad - m(\tau) \times \Lambda_\tau^{(\tau)} d\tau \end{aligned} \tag{6.75}$$

So we may continue our calculations in equation (6.74) as follow:

$$\begin{aligned}
\mathbb{E}_t\left[\int_t^T \pi_s^\xi ds + \pi_T^\xi\right] &= \int_0^T m(s)ds + m(T) \\
&\quad + \int_0^t d\left(\int_\cdot^T m(s) \times \Lambda_\cdot^{(s)} ds\right)_\tau \\
&\quad + \int_0^t d\left(m(T) \times \Lambda_t^{(T)}\right) \\
&= \int_0^T m(s)ds + m(T) - \int_0^t m(\tau) \times \Lambda_\tau^{(\tau)} d\tau \\
&\quad + \int_0^t \left(\int_\tau^T \xi m(s) \Lambda_\tau^{(s)} \left(\boldsymbol{\sigma}^{(s)}(\tau) - \boldsymbol{\lambda}(\tau)\right)^T ds\right) d\boldsymbol{W}_\tau \\
&\quad + \int_0^t \xi m(T) \Lambda_\tau^{(T)} \left(\boldsymbol{\sigma}^{(T)}(\tau) - \boldsymbol{\lambda}(\tau)\right)^T d\boldsymbol{W}_\tau \\
&= \int_0^T m(s)ds + m(T) - \int_0^t m(\tau) \times \Lambda_\tau^{(\tau)} d\tau \\
&\quad + \int_0^t \xi \left\{\int_\tau^T m(s) \Lambda_\tau^{(s)} \boldsymbol{\sigma}^{(s)}(\tau) ds + m(T) \Lambda_\tau^{(T)} \boldsymbol{\sigma}^{(T)}(\tau)\right. \\
&\qquad \left. - \mathbb{E}_\tau\left[\int_\tau^T \pi_s^\xi ds + \pi_T^\xi\right] \boldsymbol{\lambda}(\tau)\right\}^T d\boldsymbol{W}_\tau. \qquad (6.76)
\end{aligned}$$

We may conclude that:

$$\begin{aligned}
\boldsymbol{a}_t^{\left(C^{(\boldsymbol{\lambda})}, Z^{(\boldsymbol{\lambda})}, \boldsymbol{\lambda}\right)} &\triangleq \frac{\left[\boldsymbol{W}, \pi^{(\boldsymbol{\lambda})} W^{\left(C^{(\boldsymbol{\lambda})}, Z^{(\boldsymbol{\lambda})}, \boldsymbol{\lambda}\right)}\right]_t'}{\pi_t^{(\boldsymbol{\lambda})} W_t^{\left(C^{(\boldsymbol{\lambda})}, Z^{(\boldsymbol{\lambda})}, \boldsymbol{\lambda}\right)}} \\
&= \frac{\left[\boldsymbol{W}, \mathbb{E}_t\left[\int_t^T \pi_s^\xi ds + \pi_T^\xi\right]\right]_t'}{\mathbb{E}_t\left[\int_t^T \pi_s^\xi ds + \pi_T^\xi\right]} \\
&= \xi \left\{\frac{\int_t^T m(s) \Lambda_t^{(s)} \boldsymbol{\sigma}^{(s)}(t) ds + m(T) \Lambda_t^{(T)} \boldsymbol{\sigma}^{(T)}(t)}{\int_t^T m(s) \Lambda_t^{(s)} ds + m(T) \Lambda_t^{(T)}} - \boldsymbol{\lambda}(t)\right\}. \qquad (6.77)
\end{aligned}$$

By using equation (4.35) we will find the optimal portfolio of equation (6.65):

$$\begin{aligned}
\boldsymbol{u} &= \left(\Sigma\Sigma^T\right)^{-1} \boldsymbol{b} + \left(\Sigma^T\right)^{-1} \boldsymbol{a}^{\left(C^{(\boldsymbol{\lambda})}, Z^{(\boldsymbol{\lambda})}, \boldsymbol{\lambda}\right)} \\
&= (1-\xi)\left(\Sigma\Sigma^T\right)^{-1} \boldsymbol{b} \\
&\quad + \xi\left(\Sigma^T\right)^{-1} \left\{\frac{\int_t^T m(s) \Lambda_t^{(s)} \boldsymbol{\sigma}^{(s)}(t) ds + m(T) \Lambda_t^{(T)} \boldsymbol{\sigma}^{(T)}(t)}{\int_t^T m(s) \Lambda_t^{(s)} ds + m(T) \Lambda_t^{(T)}}\right\}.
\end{aligned}$$

To obtain the optimal consumption of equation (6.66), we may proceed as follow:

$$
\begin{aligned}
c_t &= \frac{\pi C_t}{\pi_t W_t^{(\eta)}} \\
&= \frac{\eta^{\frac{1}{\gamma-1}} \pi_t^{\xi}}{\eta^{\frac{1}{\gamma-1}} \left(\int_t^T m(s) \times \Lambda_t^{(s)} ds + m(T) \times \Lambda_t^{(T)} \right)} \\
&= \frac{m(t) \times \Lambda_t^{(t)}}{\int_t^T m(s) \times \Lambda_t^{(s)} ds + m(T) \times \Lambda_t^{(T)}}, \qquad (6.78)
\end{aligned}
$$

where in the last step we used equation (6.59). □

Chapter 7

Conclusions and Recommendations

In this project, we considered the classical portfolio choice problem, also known as Merton's problem, when the opportunity set is stochastic. The two main approaches to the problem, namely the stochastic control theory and the martingale (or duality) approach, were discussed along with an extended literature study in chapters 1, 3, and 4. Then we proposed a new approach to portfolio choice problem, called the *direct approach*, in chapter 5. Finally, in chapter 6, we solve three special cases: logarithmic utility in an incomplete market, power utility in the original Merton's setting, and power utility in a market with Gaussian term structure. For the logarithmic case we provided three different approaches. For the power utility cases, we find out that the optimal solutions differ in the following ways,

- In the Merton setting, the optimal portfolio only includes the myopic term, while the excess hedging demand term (containing the covariance between zero-coupon bonds and assets) enters in the optimal portfolio for the Gaussian term structure case.
- In the Merton setting the optimal portfolio is deterministic whether we consider consumption or not, but for the Gaussian term structure case the optimal portfolio is deterministic when there is no consumption, and stochastic otherwise.

As the potential line of research based on the work conducted in this project, we propose the following topics:

- Considering specific stochastic factor models
  We may consider stochastic factors to model asset predictability, stochastic volatility, and interest rates. For example one may take other term-structure models such as C.I.R. or a stochastic volatility model such as Heston model. Alternatively one may try to consider more general market settings such as quadratic asset returns (which includes both C.I.R. and Heston model, see [61]) or HJM model (for example see [73]). See chapters 1 and 3 for more references.
- Adapting the direct approach to alternative preference criteria
  As mentioned earlier, the traditional *static* choice of the utility function and investment horizon is not realistic. it has long been recognized by economists that preferences may not be Intertemporally separable. In particular, the utility associated with the choice of consumption at a given date is likely to depend on past choices of consumption. For example high past consumption generates a desire for high current consumption. According to Browning [13], this idea dates back to the 1890 book 'Principles of Economics' by

Alfred Marshall. Generalizations of standard time-separable preferences that have been suggested include *recursive* or *stochastic differential utility* (see, for example, [32, 76, 77]), *habit formation criterion* (see [67]), and *forward performance criterion* (see [68, 87]). One may try to adapt the idea of the direct approach to these alternative formulations.

- Providing validity conditions for the direct approach
  As mentioned in chapter 5, we have only checked the first order optimality conditions. So, technically, the results obtain by the direct approach can only be considered as candidates for the optimal solution. One may try to provide sufficient conditions, or ideally the minimal conditions, under which these candidate solutions are indeed optimal.

- Extending the direct approach to general semimartingale markets
  One may try to extend the direct approach to a general semimartingale market. The main tools in developing the direct approach are properties of stochastic exponentials, which can still be used in a semimartingale setting. Nonetheless, such extensions would be more involved and obtainting validity conditions would be more demanding.

Appendix A

Stochastic Exponential and Logarithms

Definition A.1. Let $\boldsymbol{X} = \left(X_t^{(1)}, \ldots, X_t^{(k)}\right)^T$ be a $k \times 1$ semimartingale. Then the stochastic exponential of $\boldsymbol{X}$, denoted by $\mathcal{E}(\boldsymbol{X})$ is defined as the unique solution of the following SDE

$$\boldsymbol{Y} = \mathbf{1} + \int_0^t \operatorname{diag}(\boldsymbol{Y}_{u-})\, d\boldsymbol{X}_u. \tag{A.1}$$

Definition A.2. Let $\boldsymbol{Y} = \left(Y_t^{(1)}, \ldots, Y_t^{(k)}\right)^T$ be a $k \times 1$ semimartingale such that, for all k, the two processes $Y^{(k)}$ and $Y_-^{(k)}$ do not vanish. Then the stochastic logarithm of $\boldsymbol{Y}$, denoted by $\mathcal{L}\mathrm{og}(\boldsymbol{Y})$, is defined as

$$\mathcal{L}\mathrm{og}(\boldsymbol{Y}) \triangleq \int_0^{\cdot} (\operatorname{diag}(\boldsymbol{Y}_{u-}))^{-1}\, d\boldsymbol{Y}_u. \tag{A.2}$$

Equivalently,$\mathcal{L}\mathrm{og}(\boldsymbol{Y})$ is the unique process $\boldsymbol{X}$ satisfying $\boldsymbol{Y} = \operatorname{diag}(\boldsymbol{Y}_0)\, \mathcal{E}(\boldsymbol{X})$.

Remark A.3. Suppose that $\boldsymbol{Y} = \operatorname{diag}(\boldsymbol{Y}_0)\, \mathcal{E}(\boldsymbol{X})$. We usually denote $d\boldsymbol{X}$ by $\frac{d\mathbf{Y}}{\mathbf{Y}}$, for example for a given predictable process $\boldsymbol{\varphi}$ we have:

$$\int_0^T \boldsymbol{\varphi}^T \frac{d\mathbf{Y}}{\mathbf{Y}} \triangleq \int_0^T \boldsymbol{\varphi}^T d\boldsymbol{X}. \tag{A.3}$$

In a similar way, we define:

$$\frac{d[\varphi, \mathbf{Y}]}{\boldsymbol{Y}} \triangleq d[\varphi, \boldsymbol{X}]. \tag{A.4}$$

Note that we have $d\boldsymbol{Y} = \operatorname{diag}(\boldsymbol{Y})\, d\boldsymbol{X}$, so we also have:

$$\begin{aligned} d\boldsymbol{X} &= (\operatorname{diag}(\boldsymbol{Y}))^{-1}\, d\boldsymbol{Y}, \\ d[\varphi, \boldsymbol{X}] &= (\operatorname{diag}(\boldsymbol{Y}))^{-1}\, d[\varphi, \boldsymbol{Y}]. \end{aligned} \tag{A.5}$$

Proposition A.4. *Let X and Y be two continuous semimartingale, and α be a constant. Then:*
(i) $\mathcal{E}(X) \times \mathcal{E}(Y) = e^{[X,Y]}\mathcal{E}(X+Y)$
(ii) $\frac{\mathcal{E}(X)}{\mathcal{E}(Y)} = e^{[Y]-[X,Y]}\mathcal{E}(X-Y)$
(iii) $\mathcal{E}(X)^{\alpha} = e^{\frac{\alpha(\alpha-1)}{2}[X]}\mathcal{E}(\alpha X)$

Proof. Note that we have:

$$\begin{aligned} d\mathcal{E}(X) &= \mathcal{E}(X)\,dX \\ d[\mathcal{E}(X)] &= d[\int \mathcal{E}(X)\,dX] = \mathcal{E}(X)^2\,d[X] \end{aligned}$$

Similar equations hold for Y. We also have:All the cases are straightforward application of Itô's formula.

(i) We have:

$$\begin{aligned} d(\mathcal{E}(X)\mathcal{E}(Y)) &= \mathcal{E}(X)\,d\mathcal{E}(Y) + \mathcal{E}(Y)\,d\mathcal{E}(X) + d[\mathcal{E}(X),\mathcal{E}(Y)] \\ &= \mathcal{E}(X)\mathcal{E}(Y)\left\{\frac{d\mathcal{E}(Y)}{\mathcal{E}(Y)} + \frac{d\mathcal{E}(X)}{\mathcal{E}(X)} + \frac{d[\mathcal{E}(X),\mathcal{E}(Y)]}{\mathcal{E}(X)\mathcal{E}(Y)}\right\} \\ &= \mathcal{E}(X)\mathcal{E}(Y)\{dY + dX + d[X,Y]\} \\ &= \mathcal{E}(X)\mathcal{E}(Y)\,d(X+Y+[X,Y]) \end{aligned}$$

So we obtain:

$$\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X+Y+[X,Y]) = e^{[X,Y]}\mathcal{E}(X+Y)$$

(ii) Again by Itô's formula:

$$\begin{aligned} d\frac{\mathcal{E}(X)}{\mathcal{E}(Y)} &= \frac{d\mathcal{E}(X)}{\mathcal{E}(Y)} - \frac{\mathcal{E}(X)}{\mathcal{E}(Y)}\frac{d\mathcal{E}(Y)}{\mathcal{E}(Y)} + \frac{1}{2}\left\{2\frac{\mathcal{E}(X)}{\mathcal{E}(Y)}\frac{d[\mathcal{E}(Y)]}{\mathcal{E}(Y)^2} - 2\frac{\mathcal{E}(X)}{\mathcal{E}(Y)}\frac{d[\mathcal{E}(X),\mathcal{E}(Y)]}{\mathcal{E}(X)\mathcal{E}(Y)}\right\} \\ &= \frac{\mathcal{E}(X)}{\mathcal{E}(Y)}\left\{\frac{d\mathcal{E}(X)}{\mathcal{E}(X)} - \frac{d\mathcal{E}(Y)}{\mathcal{E}(Y)} + \frac{d[\mathcal{E}(Y)]}{\mathcal{E}(Y)^2} - \frac{d[\mathcal{E}(X),\mathcal{E}(Y)]}{\mathcal{E}(X)\mathcal{E}(Y)}\right\} \\ &= \frac{\mathcal{E}(X)}{\mathcal{E}(Y)}\{dX - dY + d[Y] - d[X,Y]\} \end{aligned}$$

Hence:

$$\frac{\mathcal{E}(X)}{\mathcal{E}(Y)} = \mathcal{E}(X - Y + [Y] - [X,Y]) = e^{[Y]-[X,Y]}\mathcal{E}(X-Y)$$

(iii) By Itô's formula:

$$\begin{aligned} d\mathcal{E}(X)^\alpha &= \alpha\mathcal{E}(X)^{\alpha-1}\,d\mathcal{E}(X) + \frac{1}{2}\alpha(\alpha-1)\mathcal{E}(X)^{\alpha-2}\,d[\mathcal{E}(X)] \\ &= \mathcal{E}(X)^\alpha\left\{\alpha\frac{d\mathcal{E}(X)}{\mathcal{E}(X)} + \frac{\alpha(\alpha-1)}{2}\frac{d[\mathcal{E}(X)]}{\mathcal{E}(X)^2}\right\} \\ &= \mathcal{E}(X)^\alpha\left\{\alpha dX + \frac{\alpha(\alpha-1)}{2}d[X]\right\} \end{aligned}$$

Finally we obtain:

$$\mathcal{E}(X)^\alpha = \mathcal{E}\left(\alpha X + \frac{\alpha(\alpha-1)}{2}[X]\right) = e^{\frac{\alpha(\alpha-1)}{2}[X]}\mathcal{E}(\alpha X)$$

□

Appendix B

Other Results

Lemma B.1. (variant of Bay's rule)- *Let $Z = (Z_t)$ be a positive martingale with $Z_0 = 1$ under measure $\mathbb{P}$. Define a new measure $\mathbb{Q}$ by $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$.* Then a process $X = (X_t)$ is a $\mathbb{Q}$-(local) martingale if and only if ZX is a $\mathbb{P}$ (local) martingale.

Theorem B.2. (change of measure)- *Suppose, under measure $\mathbb{P}$, $Z = \mathcal{E}(Y)$ is a positive martingale for some semimartingale $Y = (Y_t)$, and define measure $\mathbb{Q}$ by $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$.* Then a process $X = (X_t)$ is a $\mathbb{Q}$-local martingale if and only if $X + [X, Y]$ is a $\mathbb{P}$-local martingale.

Proof. From lemmaB.1, X is a $\mathbb{Q}$ local martingale if and only if ZX is a $\mathbb{P}$-local martingale. But we have

$$d(ZX) = X_- dZ + Z_- dX + d[X, Z],$$

The first term in the right-hand-side is a local martingale, since Z is a martingale. So ZX is a local martingale if and only if $V \triangleq \int Z_- dX + [X, Z]$ is a local martingale. Define $U \triangleq X + [X, Y]$. Note that we have:

$$\begin{aligned} V &= \int Z_- dX + [X, Z] = \int Z_- dX + \int Z_- d[X, Y] = \int Z_- dU \\ U &= \int \frac{1}{Z_-} dV \end{aligned}$$

Hence V is a local martingale if and only if $U \triangleq X + [X, Y]$ is a local martingale. □

Theorem B.3. *(Linear SDE) Consider the following SDE:*

$$\begin{cases} d\boldsymbol{r}_t = (\boldsymbol{\theta}(t) + A(t)\boldsymbol{r}_t)\, dt + \Delta(t) d\boldsymbol{W}_t \\ \boldsymbol{r}_0 \text{ is given} \end{cases} \tag{B.1}$$

Where $\mathcal{W} = (\mathcal{W}_t)$ is a k-dimensional Brownian motion, and $A(.)$, $\boldsymbol{\theta}(.)$ and $\Delta(.)$ are deterministic matrix functions of appropriate dimensions. Then subject to some stability conditions on the coefficients, the unique strong solution of equation(B.1)is given by:

$$\boldsymbol{r}_t = \Upsilon(t) \left\{ \boldsymbol{r}_0 + \int_0^t \Upsilon^{-1}(u)\boldsymbol{\theta}(u) du + \int_0^t \Upsilon^{-1}(u)\Delta(u) d\boldsymbol{W}_u \right\}. \tag{B.2}$$

Where $\Upsilon_{k\times k}(t)$ is the solution of the following matrix differential equations:

$$\begin{cases} \dot{\Upsilon}(t) = A(t)\Upsilon(t), \\ \Upsilon(0) = I_{k\times k}. \end{cases} \tag{B.3}$$

Furthermore the mean and covariance functions of r*, namely* $\boldsymbol{m}(t) = \mathbb{E}[r_t]$ *and* $V(t) = \mathbb{E}\left[(r_t - m(t))(r_t - m(t))^T\right]$ *are the solution of the linear equations:*

$$\begin{array}{l}\dot{\boldsymbol{m}}(t) = A(t)\boldsymbol{m}(t) + \boldsymbol{\theta}(t),\\ \dot{V}(t) = A(t)V(t) + V(t)A^T(t) + \Delta(t)\Delta^T(t),\end{array} \tag{B.4}$$

which have the following solutions:

$$\begin{array}{l}\boldsymbol{m}(t) = \Upsilon(t)\left\{\boldsymbol{m}(0) + \int_0^t \Upsilon^{-1}(s)\boldsymbol{\theta}(s)ds\right\},\\ V(t) = \Upsilon(t)\left\{V(0) + \int_0^t \Upsilon^{-1}(u)\Delta(u)\left(\Upsilon^{-1}(u)\Delta(u)\right)^T du\right\}\Upsilon^T(t).\end{array} \tag{B.5}$$

Finally the auto-correlation function of $\boldsymbol{r}$*, i.e.* $\boldsymbol{\rho}(s,t) = \mathbb{E}\left[(\boldsymbol{r}_s - \boldsymbol{m}(s))(\boldsymbol{r}_t - \boldsymbol{m}(t))^T\right]$ *is given by:*

$$\boldsymbol{\rho}(s,t) = \Upsilon(s)\left\{V(0) + \int_0^{s\wedge t} \Upsilon^{-1}(u)\Delta(u)\left(\Upsilon^{-1}(u)\Delta(u)\right)^T du\right\}\Upsilon^T(t). \tag{B.6}$$

Theorem B.4. *Consider the one-factor Hull-White extension of Vasicek model:*

$$dr_t = (\vartheta(t) - ar_t))dt + \delta d\mathcal{W}_t$$

Also assume that initial bond prices $P^M(0,T)$*, and initial instantaneous forward-rate curve,* $f^M(0,T)$ *are given.Then we have:*

(i)The short-rate r_t *is explicitly given by:*

$$\begin{aligned} r_t &= r_s e^{-a(t-s)} + \int_s^t e^{-a(t-u)}\vartheta(u)du + \delta\int_s^t e^{-a(t-u)}dW_u \\ &= (r_s - \omega(s))\, e^{-a(t-s)} + \omega(t) + \delta\int_s^t e^{-a(t-u)}dW_u \end{aligned}$$

Where:

$$\omega(t) \triangleq f^M(0,t) + \frac{\delta^2}{2a^2}\left(1 - e^{-at}\right)^2$$

(ii)r_t *is Gaussian with mean and variance given by:*

$$\begin{aligned} \mathbb{E}_s[r_t] &= (r_s - \omega(s))\, e^{-a(t-s)} + \omega(t) \\ Var_s[r_t] &= \frac{\delta^2}{2a}\left(1 - e^{-2a(t-s)}\right) \end{aligned}$$

(iii) $\int_s^t r_u du|\mathcal{F}_s$ *is also Gaussian with (conditional) mean and variance given by:*

$$\begin{array}{c}\mathbb{E}_s\left[\int_s^t r_u du\right] = B(s,t)(r_s - \omega(s)) + \log\left(\frac{P^M(0,s)}{P^M(0,t)}\right) + \frac{1}{2}\left(V(0,t) - V(0,s)\right)\\ Var_s\left[\int_s^t r_u du\right] = V(s,t)\end{array} \tag{B.7}$$

where:

$$\begin{aligned} B(s,t) &\triangleq \frac{1}{a}\left(1 - e^{-a(T-t)}\right) \\ V(s,t) &\triangleq \frac{\delta^2}{a^2}\left(t - s + \frac{2}{a}e^{-a(t-s)} - \frac{1}{2a}e^{-2a(t-s)} - \frac{3}{2a}\right) \end{aligned}$$

(iv) For $s < t$ we have:

$$P(s,t) \triangleq \mathbb{E}_s\left[e^{-\int_s^t r_u du}\right] = A(s,t)e^{-B(s,t)r_s}$$

where:

$$A(s,t) = \frac{P^M(0,t)}{P^M(0,s)} e^{\left\{f^M(0,s)B(s,t) - \frac{\delta^2}{4a}\left(1-e^{-2as}\right)B^2(s,t)\right\}}$$

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